# Modelling Electroelastic Materials

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# Electroelasticity

Electroelasticity is used to model materials, such as electro-active polymers, that deform elastically in an electric field.

In order to model the interaction between electrostatics and non-linear elasticity, we focus on a deformable and electrically polarizable material.

The electric field, $\mathbf{E}$, and the electric displacement, $\mathbf{D}$, satisfy the following simplification of Maxwell's Equations in polarized materials,

$$\text{curl}\mathbf{E} = \mathbf{0} \quad \text{div}\mathbf{D} = \rho_f \tag{1}$$

where $\rho_f$ is the free charge density of the material.

# Total Cauchy Stress

The equilibrium equation for nonlinear electroelastic interactions can be written in a similar form to the corresponding nonlinear elastic equation.

$$\text{div}\boldsymbol{\sigma} + \rho\mathbf{f} = \mathbf{0} \tag{2}$$

By adding an electric body force term of the form $\text{div}\boldsymbol{\tau}_e$, where

$$\boldsymbol{\tau}_e = \mathbf{D} \otimes \mathbf{E} - \frac{1}{2}\epsilon_0(\mathbf{E} \cdot \mathbf{E})\mathbf{I} \tag{3}$$

we get an equilibrium equation of the form

$$\text{div}\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0} \tag{4}$$

where $\boldsymbol{\tau} = \boldsymbol{\sigma} + \boldsymbol{\tau}_e$, is the total Cauchy stress tensor.

# Lagrangian Formulation

There are many possible formulations for the stress tensors and energy balance laws. The formulation based on a 'total stress' avoids the need to define either an electric body force or Maxwell stress ($\boldsymbol{\tau}_e$) in the material.

Lagrangian formulations, which are based on reference values of the electric field and electric displacement, lead to constitutive laws and governing equations similar to those of nonlinear elasticity.

# Lagrangian Formulation

The Lagrangian electric field and electric displacement are defined by,

$$\mathbf{E}_{\mathrm{L}} = \mathbf{F}^{\mathrm{T}}\mathbf{E} \quad \mathbf{D}_{\mathrm{L}} = J\mathbf{F}^{-1}\mathbf{D} \tag{5}$$

where $\mathbf{F}$ is the deformation gradient and $J = \det\mathbf{F}$.

We can then define the total energy density function in two different ways, with either $\mathbf{E}_{\mathrm{L}}$ or $\mathbf{D}_{\mathrm{L}}$ as the independent electric variable, denoted,

$$\Omega = \Omega(\mathbf{F}, \mathbf{E}_{\mathrm{L}}) \text{ and } \Omega^* = \Omega^*(\mathbf{F}, \mathbf{D}_{\mathrm{L}}) \tag{6}$$

# Lagrangian Formulation

These energy equations and the expressions for the Lagrangian electric variables give the following constitutive equations for the stress tensor and electric displacement / electric field respectively.

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} \quad \mathbf{D} = -J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{E}_{\mathrm{L}}} \quad \mathbf{E} = \mathbf{F}^{-\mathrm{T}}\frac{\partial \Omega^*}{\partial \mathbf{D}_{\mathrm{L}}} \tag{7}$$

Incompressibility is an important condition for electro-sensitive elastomers. When this condition is imposed, it introduces a Lagrange multiplier so that,

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I} \tag{8}$$

# Isotropic Materials

If the material is isotropic, $\Omega$ ($\Omega^*$) depends on six invariants, the principal invariants of $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and three independent invariants that depend on $\mathbf{E}_{\mathrm{L}}$ ($\mathbf{D}_{\mathrm{L}}$).

A possible choice of invariants that depend on $\mathbf{E}_{\mathrm{L}}$ are, $I_4 = \mathbf{E}_{\mathrm{L}} \cdot \mathbf{E}_{\mathrm{L}}$, $I_5 = (\mathbf{c}\mathbf{E}_{\mathrm{L}}) \cdot \mathbf{E}_{\mathrm{L}}$ and $I_6 = (\mathbf{c}^2\mathbf{E}_{\mathrm{L}}) \cdot \mathbf{E}_{\mathrm{L}}$, which, for an incompressible material, give

$$\begin{aligned} \boldsymbol{\tau} = 2\Omega_1\mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} \\ + 2\Omega_5\mathbf{bE}\otimes\mathbf{bE} + 2\Omega_6(\mathbf{bE}\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes\mathbf{bE}) \end{aligned} \quad (9)$$

$$\mathbf{D} = -2(\Omega_4\mathbf{b} + \Omega_5\mathbf{b}^2 + \Omega_6\mathbf{b}^3)\mathbf{E}$$

# Example: Equibiaxial

We consider an incompressible rectangular slab under an equibiaxial deformation given by $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = \lambda^{-2}$ and deformation gradient $\mathbf{F}$ with stretches along the diagonal.

We apply a voltage to the thickness direction of the slab, i.e. 'direction 3', so that the electric field is given by $\mathbf{E} = (0, 0, E_3)$.

We notice that all of the independent variables will depend only on $\lambda$ and $E_L$, the magnitude of $\mathbf{E}_{\mathrm{L}}$. We can therefore write $\omega = \omega(\lambda, E_L) = \Omega(\mathbf{F}, \mathbf{E}_{\mathrm{L}})$.

# Example: Equibiaxial

We can then write equations (9) in terms of derivatives of the function $\omega$ to give the equations,

$$\tau_{11} = \tau_{22} = \frac{1}{2}\lambda\omega_\lambda$$
$$D_{L3} = -\omega_{E_L}$$

which are much easier to solve given an energy density function $\omega$.

These equations can then be used to model electro-sensitive elastomers undergoing equibiaxial deformations.

# References


- Dorfmann, L., Ogden, R.W., *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, Springer, 2014.
- Zhao, X., Suo, Z., *Method to analyze electromechanical stability of dielectric elastomers*, Applied Physics Letters, 91, 2007