# Counting the number of Entry pattern matrices

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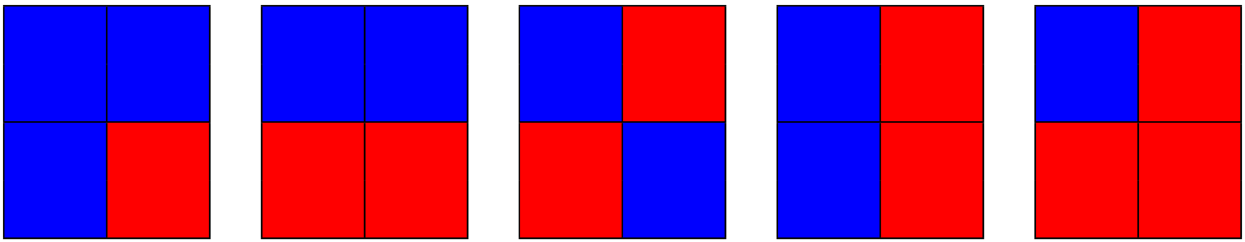
# Example

Assume we have an $n \times n$ chessboard. And we want to color the squares by $k$ colors (each color occurs at least once); and two colourings are different only if one can not be obtained from another by swaps on rows or columns (or both). For $n = 2$, there are 5 such colourings with blue and red:



Let $x = "blue", y = "red"$. Then the above colourings correspond to the following matrices:

$$\begin{bmatrix} x & x \\ x & y \end{bmatrix}, \begin{bmatrix} x & x \\ y & y \end{bmatrix}, \begin{bmatrix} x & y \\ y & x \end{bmatrix}, \begin{bmatrix} x & y \\ x & y \end{bmatrix}, \begin{bmatrix} x & y \\ y & y \end{bmatrix}$$

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- Each entry is an element of a specified set of independent indeterminates.
- Entries can be the same, but can not be a constant.

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| $\begin{bmatrix} x & x \\ x & y \end{bmatrix}, \begin{bmatrix} x & y \\ z & t \end{bmatrix}, \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}$ | $\begin{bmatrix} x & 0 \\ x & y \end{bmatrix}, \begin{bmatrix} x & y \\ 2x & y \end{bmatrix}$ |

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*This talk is for **counting the number of** of entry pattern matrices (EPMs for short).*

# The number of EPMs

Let $A(x_1, \cdots, x_k)$ be an $m \times n$ EPM with $k$ indeterminates, each indeterminate occurs at least once in the matrix. Let $a_i$ be the number of $x_i$ in the matrix. Then

1. $a_1 + a_2 + \cdots + a_k = mn, a_i \geq 1$,
2. $\binom{mn}{a_1}$ is the number of ways to choose entries to put $x_1$ in,
3. $\binom{mn - a_1}{a_2}$ is the number of ways to choose entries to put $x_2$ in.
4. and so on.

Hence, the number of EPMs $A(x_1, \cdots, x_k)$ of degree $m \times n$ is

$$\Delta_{m,n,k} := \sum_{\substack{a_1 + \cdots + a_k = mn \\ a_i \geq 1}} \binom{mn}{a_1} \binom{mn - a_1}{a_2} \cdots \binom{mn - a_1 - a_2 - \cdots - a_{k-1}}{a_k}$$

So,

$$\Delta_{m,n,k} = \sum_{\substack{a_1 + \cdots + a_k = mn \\ a_i \geq 1}} \frac{(mn)!}{a_1! a_2! \cdots a_k!}$$

For $k = 2$:

$$\Delta_{m,n,2} = \sum_{\substack{a_1+a_2=mn \\ a_i \geq 1}} \frac{(mn)!}{a_1!a_2!} = \sum_{\substack{a_1+a_2=mn \\ a_i \geq 0}} \frac{(mn)!}{a_1!a_2!} - 2 \sum_{\substack{a_1+a_2=mn \\ a_1 = 0}} \frac{(mn)!}{a_1!a_2!}$$

$$\Rightarrow \Delta_{m,n,2} = 2^{mn} - 2$$

Similarly,

$$\Delta_{m,n,3} = 3^{mn} - 3\Delta_{m,n,2} - 3 = 3^{mn} - 3 \cdot 2^{mn} + 3$$

$$\Delta_{m,n,k} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^{mn}$$

$$A_{m\times n}(x_1,\cdots,x_k) \sim B_{m\times n}(x_1,\cdots,x_k) \Leftrightarrow PAQ^t = B,$$

where $P, Q$ are permutation matrices.
The number of equivalent classes under this relation is equal to the number of orbits under the action of the group $G := \mathbb{P}_m \times \mathbb{P}_n$ on the set $X := M_{m\times n}(x_1,\cdots,x_k)$. By Burnside's lemma, it is equal to

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where $X^g = \{x \in X : gx = x\}$ is the set of points fixed by $g$.

Now let $\sigma = \sigma_1\sigma_2\cdots\sigma_p$ and $\delta = \delta_1\delta_2\cdots\delta_q$, then the set of entries of $X$ splits into $p \times q$ blocks, the first block is the block corresponding to the rows in $\sigma_1$ and the columns in $\delta_1$, and so on. Let's denote $l(\sigma)$ for the length of cycle $\sigma$ and $A_1$ be the first $m_1 \times n_1$ block . Then $A_1$ is fixed by $\sigma_1$ and $\delta_1$ if and only if

$$(A_1)_{ij} = (A_1)_{\sigma_1(i)\delta_1(j)} = (A_1)_{\sigma_1^2(i)\delta_1^2(j)} = (A_1)_{\sigma_1^3(i)\delta_1^3(j)} = \cdots$$

The length of this equality is

$$gcd(l(\sigma_1), l(\delta_1)).$$

Hence,

$$|X^{\sigma,\delta}| = \sum_{\sum \frac{l(\sigma_i)\cdot l(\delta_j}{gcd(l(\sigma_i),l(\delta_j))} \geq k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^{\sum \frac{l(\sigma_i)\cdot l(\delta_j}{gcd(l(\sigma_i),l(\delta_j))}}$$

# Example

1. $m = n = 2, k = 2.$
   There are $2^4 - 2 = 14$ such EPMs and 5 classes.

$$\begin{bmatrix} x & x \\ x & y \end{bmatrix} \sim \begin{bmatrix} x & y \\ x & x \end{bmatrix} \sim \begin{bmatrix} y & x \\ x & x \end{bmatrix} \sim \begin{bmatrix} x & x \\ y & x \end{bmatrix}$$

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$$\begin{bmatrix} y & y \\ x & y \end{bmatrix} \sim \begin{bmatrix} y & y \\ y & x \end{bmatrix} \sim \begin{bmatrix} y & x \\ y & y \end{bmatrix} \sim \begin{bmatrix} x & y \\ y & y \end{bmatrix}$$

# Example (cont.)

In this case, the formula gives

$$\frac{1}{|G|}\sum_{g\in G}|X^g| = \frac{1}{4}(|X^{id,id}| + |X^{id,(12)}| + |X^{(12),id}| + |X^{(12),(12)}|)$$

$$= \frac{1}{4}\left((2^4-2)+(2^2-2)+(2^2-2)+(2^2-2)\right) = 5$$

THANK YOU!