Introduction to Modular Forms

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Introduction

In this talk we will discuss modular forms and their number-theoretic properties. We will also examine the connection between this area of mathematics and that of VOA theory.

The Modular Group

Consider the set of 2×2 matrices with determinant 1. We note that it is closed under multiplication, and hence forms a group. This group is known as the *special linear group* of degree 2 over the integers or the *modular group* and is denoted by $SL(2, \mathbb{Z})$. We will also note here that $SL(2, \mathbb{Z})$ is generated by the matrices:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Modular Transformations

We define the *upper half-plane* $\mathfrak{H}$ of complex numbers to be the set $\{z \in \mathbb{C} : \Im(z) > 0\}$. Then we can define a group action of $SL(2, \mathbb{Z})$ on $\mathfrak{H}$ by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, $\tau \in \mathfrak{H}$. Then $S \cdot \tau = -\frac{1}{\tau}$, $T \cdot \tau = \tau + 1$

Modular Forms

We now define a modular form. A modular form a function $f(\tau)$ which

- Is holomorphic on the upper half-plane
- Satisfies
$$\left(\frac{a\tau + b}{cz + d}\right) = (c\tau + d)^k f(\tau) \tag{1}$$
where k is an integer known as the *weight* of the form
- Has a Fourier expansion at infinity:
$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n$$
where $q = e^{2\pi i \tau}$. These a_ns turn out to have very interesting properties.

Some Extra Notes

- Equation 1 is equivalent to:

$$f(\tau + 1) = f(\tau), \quad f\left(-\frac{1}{\tau}\right) = z^k f(\tau)$$

- The product of a modular form of weight k and one of weight k' is a form of weight $k + k'$

Some Examples

The classical example of a modular form is the *Eisenstein series*. The Eisenstein series of weight k is defined as

$$G_k(\tau) = \sum_{m,n \neq 0} \frac{1}{(m\tau + n)^k}$$

for $k \geq 4$. Note k is necessarily even. G_4 and G_6 also form a basis for the space of modular forms of weight k:

$$\mathcal{M}_k = \langle G_4^a G_6^b : 4a + 6b = k \rangle$$

For example: $\Delta(\tau) = (60G_4)^3 - 27(140G_6)^2$ Then Δ is a modular form of weight 12. We can write Δ as

$$(2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=0}^{\infty} \tau(n) q^n$$

where the $\tau(n)$ satisfy interesting number theoretic identities.

More examples

If we normalise G_k by a factor of $2\zeta(k)$ where ζ is the Riemann zeta function, then we can write

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor function, B_k is the kth Bernoulli number (coefficients of the Taylor series of $\frac{t}{e^t-1}$)
Lastly, discarding the 2π factor of Δ and taking the 24th root, we get the function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

which has weight $\frac{1}{2}$ but is not quite a modular form.

The Partition Function

Then taking $1/\eta$ we get

$$q^{-1/24}\prod_{n=1}^{\infty}\frac{1}{1-q^n} = q^{-1/24}\prod_{n=1}^{\infty}\left(\sum_{n=0}^{\infty} q^{nk}\right)$$

$$= q^{-1/24}(1+q+q^2+\cdots)(1+q^2+q^4+\cdots)(1+q^3+q^6+\cdots)\cdots$$

$$= \sum_{n=0}^{\infty} p(n) q^{n-1/24}$$

where $p(n)$ is the number of integer partitions of n.

The VOA Connection

Recall the definition of a VOA: a quadruple $(V, Y, \mathbf{1}, \omega)$ with where the following axioms hold for all $u, v \in V$:

- $$\left.\begin{aligned} L_{-1}\mathbf{1} &= 0 \\ Y(\mathbf{1}, z)u &= u \\ Y(u, z)\mathbf{1} &= u + \mathcal{O}(z) \end{aligned}\right\} \text{(vacuum)}$$

- $[L_{-1}, Y(u, z)] = \partial_z Y(u, z)$ (translation covariance)

- $(z - w)^N [Y(u, z), Y(v, w)]$, for some positive integer N (locality)

The VOA Connection

- $$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

  where L_n satisfies the Virasoro Lie algebra

  $$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}C$$

  where C is a constant called the central charge and $\delta_{m,-n}$ is the Kronecker delta
- L_0 induces a $\mathbb{Z}$-grading on V: i.e. $V = \bigoplus_{n \in \mathbb{Z}} V_n$ where $\dim V_n < \infty$ and $L_0 u = nu$ for all $u \in V_n$

The Partition Function Revisited

We define the partition function for a VOA as follows:

$$Z(q) = \mathrm{Tr}_V(q^{L(0)-c/24}) = \mathrm{Tr}_{\bigoplus_{n\geq 0} V_n}(q^{L(0)-c/24}) = \sum_{n\geq 0} q^{n-\frac{1}{24}} \mathrm{Tr}(Id_{V_n})$$

$$= q^{-\frac{1}{24}} \sum_{n\geq 0} \dim V_n q^n$$

The Heisenberg Partition Function

Take the Heisenberg VOA from the last talk ($[a(m), a(n)] = m\delta_{m,-n}$): For each $v \in V_n$ can decompose v into $a(-1)^{k_1} a(-2)^{k_2} \cdots a(-r)^{k_r} \mathbf{1}$. Then the weight of v is

$$1 \cdot k_1 + 2 \cdot k_2 + \cdots + r \cdot k_r = n$$

So dim V_n is the amount of ways we can sum an arbitrary amount of positive integers to get get n, i.e. $p(n)$. Then we have that

$$Z(q) = q^{-\frac{1}{24}} \sum_{n \geq 0} p(n) q^n = 1/\eta$$

- J.-P. Serre, A Course in Arithmetic, Springer; 1973.
- G. Mason and M.P. Tuite, *Vertex operators and modular forms*, MSRI Publications **57** 183-278 (2010), A Window into Zeta and Modular Physics, eds. K. Kirsten and F. Williams, Cambridge University Press, (Cambridge, 2010).