Genus Two n-point Functions for VOAs II

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Last time...

In the previous talk, we discussed genus one and (briefly) two Zhu n-point functions for VOAs. In this talk we will discuss the genus two details in more depth, and examine a genus two Zhu recursion formula due to Gilroy and Tuite.

We will begin with a brief recap of some relevant concepts.

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- A Virasoro vector $\omega \in V$

This data consists of the following axioms:

- For all u, v in V, we have:

$$(z-w)^N[Y(u,z),Y(v,w)]=0$$

where $[,]$ is the commutator defined by:

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- $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$ where the $L(n)$ operators satisfy the Virasoro Lie algebra:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m,-n}c$$

  where c is a constant known as the central charge.

VOAs continued

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- $Y(L(-1)v, z) = \frac{d}{dz} Y(v, z)$

Elliptic Weierstrass Functions

The classical elliptic Weierstrass functions are given by:

$$P_n(z,\tau) = \frac{1}{z^n} + (-1)^n \sum_{k=2}^{\infty} \binom{k-1}{n-1} E_k(\tau) z^{k-n}$$

for $z \in \mathbb{C}$,$\tau \in \mathbb{H}_1$, where

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n=0}^{\infty} \sigma_{k-1}(n) q^n$$

is the classical Eisenstein series (a modular form of weight k, non-trivial for even k), where $q = \exp(2\pi i \tau)$, B_k is a Bernoulli number and $\sigma_{k-1}(n)$ is the divisor function $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

n-point Functions for VOAs

We now define a genus one n-point function for a VOA by:

$$Z_V^{(1)}(v_1, z_1; \ldots; v_n, z_n; \tau)$$
$$= Tr(Y(q_1^{L(0)} v_1, q_1) \cdots Y(q_n^{L(0)} v_n, q_n) q^{L(0)-c/24})$$

where $q_i = \exp(z_i) = \sum_{n \geq 0} \frac{z_i^n}{n!}$ is a formal series in z_i.

Zhu Recursion

Zhu developed a recursion formula relating genus one n-point functions to $(n-1)$-point functions:

$$Z_V^{(1)}(v, z; v_1, z_1; \dots; v_n, z_n; \tau)$$
$$= Tr_V\left(o(v) Y(q_1^{L(0)} v_1, q_1) \cdots Y(q_n^{L(0)} v_n, q_n) q^{L(0)-c/24}\right)$$
$$+ \sum_{k=1}^{n} \sum_{j \geq 0} P_{1+j}(z - z_k, \tau) Z_V^{(1)}(v_1, z_1; \dots; v[j] v_k, z_k; \dots; v_n, z_n; \tau)$$

where $o(v) = v(wt(v) - 1)$ and $v[j]$ is the coefficient of z^{-j-1} in $Y[v, z] = Y(q_z^{L(0)} v, q_z - 1)$ with $q_z = \exp(z)$. There exists an analogue for a vertex operator super algebra (VOSA), incorporating group elements and less strict gradings - next talk.

Genus Two

The idea is to use a sewing scheme introduced by Yamada and expanded on by Mason and Tuite to develop a genus two version of the above. A genus two surface will be constructed from genus one data.

A Sewing Scheme - The ϵ-formalism

More precisely, each torus $\mathcal{S}_a = \mathbb{C}/\Lambda_a$ for $a = 1, 2$, has an associated lattice $\Lambda_a = 2\pi i(\mathbb{Z}\tau_a \oplus \mathbb{Z})$. These lattices have a minimum distance $D(\Lambda_a)$, with $\tau_a \in \mathbb{H}_1$. For local coordinate $z_a \in \mathbb{C}/\Lambda_a$, we can construct a closed disc $|z_a| \leq r_a$ which is contained in $\mathcal{S}_a$, provided

$$r_a < \frac{1}{2}D(\Lambda_a)$$

We then introduce a complex "sewing parameter" ϵ, with

$$|\epsilon| < \frac{1}{4}D(\Lambda_1)D(\Lambda_2)$$

From each surface $\mathcal{S}_a$ we excise the disc:

$$\{z_a, |z_a| \leq |\epsilon|/r_{\bar{a}}\}$$

with the convention $\bar{1} = 2$, $\bar{2} = 1$.

The ϵ-formalism

We then obtain two disjoint surfaces $\widehat{\mathcal{S}}_1$, $\widehat{\mathcal{S}}_2$, with $\widehat{\mathcal{S}}_a$ defined by:

$$\widehat{\mathcal{S}}_a = \mathcal{S}_a \backslash \{z_a, |z_a| \leq |\epsilon|/r_{\bar{a}}\}$$

Define then, the annular region:

$$\mathcal{A}_a = \{z_a, |\epsilon|/r_{\bar{a}} \leq |z_a| \leq r_a\} \subset \widehat{\mathcal{S}}_a$$

Identify $\mathcal{A}_1$, $\mathcal{A}_2$ using the sewing relation

$$z_1 z_2 = \epsilon$$

Then the new genus two surface is parametrised by

$$\mathcal{D}^\epsilon = \left\{ (\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} : |\epsilon| < \frac{1}{4} D(\Lambda_1) D(\Lambda_2) \right\}$$

The ϵ-formalism

Pictorially, this looks like

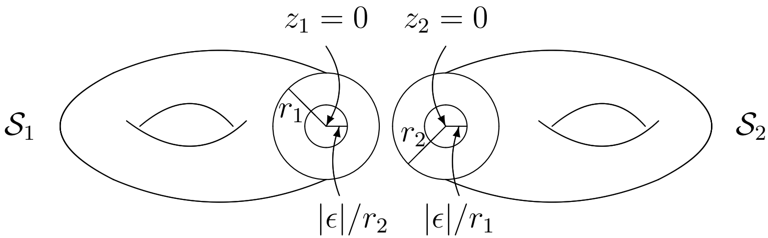

Fig. 1 Sewing Two Tori

We will refer to $\widehat{\mathcal{S}}_1$ and $\widehat{\mathcal{S}}_2$ as the left and right torus respectively.

Genus Two n-point Functions

The n-point function for a genus two VOA is then defined as

$$Z_V^{(2)}(v, x; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}, \tau_1, \tau_2, \epsilon)$$

$$= \sum_{u \in V} Z_V^{(1)}(Y[v, x]\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}]u, \tau_1) Z_V^{(1)}(\boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}]\overline{u}, \tau_2)$$

$$= \sum_{n \geq 0} \epsilon^n \sum_{u \in V_{[n]}} Z_V^{(1)}(Y[v, x]\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}]u, \tau_1) Z_V^{(1)}(\boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}]\bar{u}, \tau_2)$$

where a_l, b_r are vectors $\boldsymbol{a_l}, \boldsymbol{x_l} := a_1, x_1; \ldots; a_L, x_L$, $\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}] = Y[a_1, x_1] \cdots Y[a_L, x_L]$ etc., with the states a_l inserted on the left torus and b_r on the right, and the sum is over a basis $\{u\}$ for V.

Genus Two Zhu Recursion

A genus two Zhu recursion formula was recently introduced by Gilroy and Tuite:

$$\begin{aligned} Z_V^{(2)}(v,x;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r}) = {}& {}^N\mathcal{F}_1(x)O_1(v;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r}) \\ &+ {}^N\mathcal{F}_2(x)O_2(v;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r}) \\ &+ {}^N\mathcal{F}^{\Pi}(x)\mathbb{X}_1^{\Pi}(v;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r}) \\ &+ \sum_{l=1}^{L}\sum_{j\geq 0} {}^N\mathcal{P}_{1+j}(x,x_l)Z_V^{(2)}(\cdots;v[j]a_l,x_l;\cdots) \\ &+ \sum_{r=1}^{R}\sum_{j\geq 0} {}^N\mathcal{P}_{1+j}(x,y_r)Z_V^{(2)}(\cdots;v[j]a_l,y_r;\cdots) \end{aligned}$$

Genus Two Zhu Recursion

where

$$O_1(v; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r})$$

$$= \sum_{u \in V} Tr_V \left(o(v) \boldsymbol{Y}(\boldsymbol{q_{x_l}^{L(0)}} \boldsymbol{a_l}, \boldsymbol{q_{x_l}}) Y(q_0^{L(0)} u, q_0) q^{L(0)-c/24} \right) Z_V^{(1)}(\boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}] \overline{u}; \tau_2)$$

$$O_2(v; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r})$$

$$= \sum_{u \in V} Z_V^{(1)}(\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}] u; \tau_1) Tr_V \left(o(v) \boldsymbol{Y}(\boldsymbol{q_{y_r}^{L(0)}} \boldsymbol{b_r}, \boldsymbol{q_{y_r}}) Y(q_0^{L(0)} u, q_0) q^{L(0)-c/24} \right)$$

i.e. higher genus $o(v)$ terms. Likewise the ${}^N\mathcal{P}_{1+j}$ functions are the higher genus analogues of the P_{1+j} from genus one Zhu recursion.

Genus Two Objects

The ${}^N\mathcal{F}_a(x)$ terms are defined by

$$ {}^N\mathcal{F}_a(x) = \begin{cases} 1 + \epsilon^{1/2}\left({}^N\mathbb{Q}(x)\widetilde{\Lambda}_{\bar{a}}\right)(1), & \text{for } x \in \widehat{\mathcal{S}}_a, \\ (-1)^N\epsilon^{1/2}\left({}^N\mathbb{Q}(x)\right)(1), & \text{for } x \in \widehat{\mathcal{S}}_{\bar{a}}, \end{cases} $$

with $\mathrm{wt}[v] = N$ and ${}^N\mathbb{Q}(x)$ is an infinite row vector defined by

$$ {}^N\mathbb{Q}(x) = \mathbb{R}(x)\Delta\left(\mathbb{1} - \widetilde{\Lambda}_{\bar{a}}\widetilde{\Lambda}_a\right)^{-1} $$

with $\mathbb{R}(x)$ an infinite row vector with entries defined by:

$$ \mathbb{R}(x;m) = \epsilon^{\frac{m}{2}} P_{m+1}(x, \tau_a) $$

and Δ is an infinite matrix with entries given by

$$ \Delta(k,l) = \delta_{k,l+2N-2} $$

where $\delta_{a,b}$ is the Kronecker delta.

Genus Two Objects contd.

The $\left(\mathbb{1} - \widetilde{\Lambda}_{\bar{a}}\widetilde{\Lambda}_a\right)^{-1}$ matrix (where $\mathbb{1}$ is the infinite identity matrix) is defined as

$$\left(\mathbb{1} - \widetilde{\Lambda}_{\bar{a}}\widetilde{\Lambda}_a\right)^{-1} = \sum_{n\geq 0} (\Lambda_{\bar{a}}\Lambda_a)^n$$

with Λ_a an infinite matrix given by

$$\Lambda_a(m,n) = \epsilon^{\frac{m+n}{2}}(-1)^{n+1}\binom{m+n-1}{n} E_{m+n}(\tau_a)$$

and $\widetilde{\Lambda}_a$ is given by $\Lambda_a\Delta$.
The ${}^{N}\mathcal{F}^{\Pi}(x)$ object is a vector given by

$$\left(\mathbb{R}(x) + {}^{N}\mathbb{Q}(x)\left(\widetilde{\Lambda}_{\bar{a}}\Lambda_a + \Lambda_a\Gamma\right)\right)\Pi$$

with infinite matrices Γ, Π given by

$$\Gamma(k,l) = \delta_{k,-l+2N-2}, \quad \Pi = \Gamma^2$$

Π is a $(2N-3)$-dimensional projection matrix.

The $\mathbb{X}_a^{\Pi}$ vectors are given by $\Pi\mathbb{X}_a$, with the entries of $\mathbb{X}_a$ given by

$$\mathbb{X}_1(m) = \epsilon^{-\frac{m}{2}} \sum_{u \in V} Z_V^{(1)}(\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}]v[m]u; \tau_1) Z_V^{(1)}(\boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}]\overline{u}; \tau_2)$$

$$\mathbb{X}_2(m) = \epsilon^{-\frac{m}{2}} \sum_{u \in V} Z_V^{(1)}(\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}]u; \tau_1) Z_V^{(1)}(\boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}]v[m]\overline{u}; \tau_2)$$

Lastly, the higher genus Weierstrass functions are given by:
Define ${}^{N}\mathcal{P}_1(x,y) = {}^{N}\mathcal{P}_1(x,y;\tau_1,\tau_2,\epsilon)$ for $N \geq 1$ by

$$\begin{aligned}{}^{N}\mathcal{P}_1(x,y) =& P_1(x-y,\tau_a) - P_1(x,\tau_a)\\ &- {}^{N}\mathbb{Q}(x)\widetilde{\Lambda}_{\bar{a}}\,\mathbb{P}_1(y) - \pi_N\left({}^{N}\mathbb{Q}(x)\Lambda_{\bar{a}}\right)(K),\end{aligned}$$

for $x,y \in \widehat{\mathcal{S}}_a$ and

$$\begin{aligned}{}^{N}\mathcal{P}_1(x,y) =& (-1)^{N+1}\Big[{}^{N}\mathbb{Q}(x)\mathbb{P}_1(y) + \pi_N\epsilon^{K/2}P_{K+1}(x)\\ &+ \pi_N\left({}^{N}\mathbb{Q}(x)\widetilde{\Lambda}_{\bar{a}}\Lambda_a\right)(K)\Big],\end{aligned}$$

for $x \in \widehat{\mathcal{S}}_a,\ y \in \widehat{\mathcal{S}}_{\bar{a}}$, where $\pi_N = 1 - \delta_{N1}$ and $K = 2N-2$.

with $\mathbb{P}_{1+j}(x)$ an infinite column vector given by

$$\mathbb{P}_{1+j}(x;m) = \epsilon^{\frac{m}{2}} \binom{m+j-1}{j} \left(P_{m+j}(x,\tau_a) - \delta_{j0}E_m(\tau_a)\right)$$

Note $\mathbb{P}_{1+j}(x) = \frac{(-1)^j}{j!}\partial_x^j \mathbb{P}_1(x)$. For $j>0$ define

$$\begin{aligned} &{}^{N}\mathcal{P}_{1+j}(x,y) \\ &= \frac{1}{j!}\partial_y^j \left({}^{N}\mathcal{P}_1(x,y)\right) \\ &= \begin{cases} P_{1+j}(x-y,\tau_a) + (-1)^{1+j}.\, {}^{N}\mathbb{Q}(x)\widetilde{\Lambda}_{\bar{a}}\, \mathbb{P}_{1+j}(y), & \text{for } x,y \in \widehat{\mathcal{S}}_a, \\ (-1)^{N+1+j}.\, {}^{N}\mathbb{Q}(x)\mathbb{P}_{1+j}(y), & \text{for } x \in \widehat{\mathcal{S}}_a,\ y \in \widehat{\mathcal{S}}_{\bar{a}}. \end{cases} \end{aligned}$$

These ${}^{N}\mathcal{P}_{1+j}(x,y)$ objects are genus two analogues of the classical Weierstrass functions featured in the Zhu recursion formula for genus one VOAs.

The objective of my project is to develop a Zhu recursion formula for vertex operator super algebras (loosely speaking, VOAs with a parity) and their modules, which incorporate the action of group elements. This new formula involves “twisted” versions of the ${}^{N}\mathcal{P}_{1+j}(x,y)$ functions. More on this in the next talk.

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