Sparse and Tight Graphs in Rigidity Theory

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An Overview

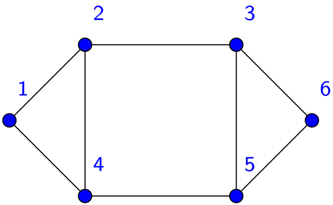
- Rigidity theory tries to answer questions of the form: given a structure defined by geometric constraints on a set of objects, what information about its geometric behaviour is implied by the underlying combinatorial structure.
- There are many class of structures have been studied but the most common one is the **bar-joint frameworks**.
- bar and joint framework is made of fixed-length bars connected by universal joints with full rotational degree of freedom. The allowed motions preserve the lengths and connectivity of the bars.

What is a framework

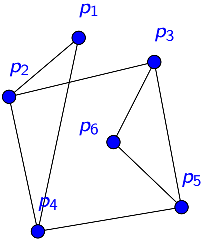
A framework $\mathcal{F}$ in $\mathbb{R}^d$ is a pair (G, P) where $G = (V, E)$ is a graph and P is a map

$$P : V \to \mathbb{R}^d \text{ where } P(i) = p_i$$

such that $p_i \neq p_j$ whenever $ij \in E$.



Graph G

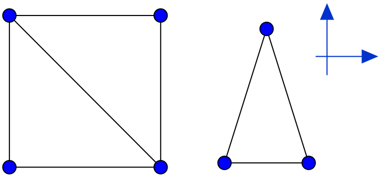


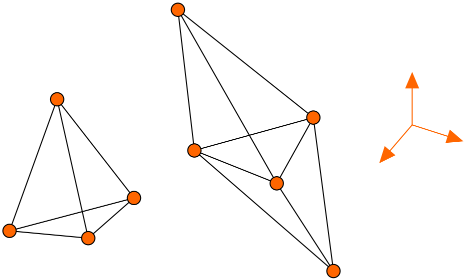
A geometric embedding

A framework in $\mathbb{R}^2$

Rigid Frameworks

- A motion of a framework $\mathcal{F} = (G, P)$ is a motion of the vertices which preserves the distance between adjacent vertices.
- A framework $\mathcal{F} = (G, P)$ is **rigid** if the only motions which it admits arise from congruences.





Rigid frameworks in $\mathbb{R}^2$..................Rigid frameworks in $\mathbb{R}^3$

(k,l)-sparse and (k,l)-tight graphs

Let $G = (V, E)$ be a graph with $|V| = v$ and $|E| = e$. Then

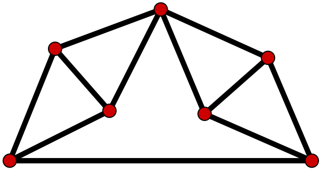
- G is $(k, l)-$sparse if for all subgraphs on v' vertices and e' edges with at least one edge and let k and l are non-negative integer numbers,
$$e' \leq kv' - l$$
- G is $(k, l)-$tight if it is $(k, l)-$sparse and $e = kv - l$.
- A freedom number of a graph is a function f^G from a graph G to $\mathbb{Z}$ defined by
$$f^G(G) = 3v - e$$

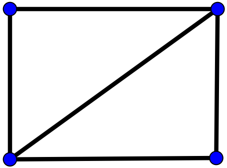
(k,l)-sparse and (k,l)-tight graphs

- A simple graph G is $(1,1)-$tight if and only if G is a tree.
- A simple connected graph G is $(1,0)-$tight if and only if G contains exactly one cycle.
- If G is a $(2,3)$-tight then it is $2-$connected.

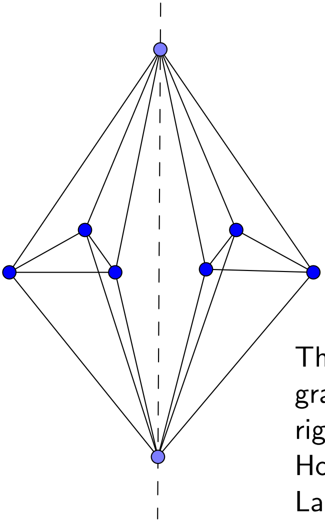
Laman's Theorem in Two Dimensional Space

- A framework $\mathcal{F} = (G, P)$ is minimally rigid if it is rigid and for every edge $e \in G$ the framework $\mathcal{F}' = (G - e, P)$ is not rigid.
- A framework $\mathcal{F} = (G, P)$ is minimally rigid if and only if its underlying graph $G = (V, E)$ is $(2, 3)$-tight.





Laman's Theorem is not enough in three dimensional space!!



The Double Banana graph is not generically rigid in dimenstion three. However it satisfies Laman's condition, i.e. it is (3,6)-tight.

Sparsity and Tightness still can Work within 3 dimensional Space! (Special Cases)

- A surface graph is a graph that is the 1-skeleton of a triangulation of a compact surface Σ (possibly with non-empty boundary).
- Theorem (Gluck 1975)
  A surface graph whose underlying surface is the sphere is 3-rigid if and only if it is $(3, 6)-$tight.
- Theorem (Cruickshank, Kitson and Power 2015)
  A surface graph whose underlying surface is the torus with a single disc removed is 3-rigid if and only if it is $(3, 6)-$tight.

Face Graphs and Block and Hole Graphs

- A face graph G is obtained from the graph of a triangulated sphere $\mathcal{S}$ by:
  1. choosing a collection of internally disjoint simplicial discs in $\mathcal{S}$.
  2. removing the edge interiors of each of these simplicial discs.
  3. labelling the non-triangular faces of the resulting planar graph by either B or H.
- A block and hole graph on a face graph G is a graph $\hat{G}$ of the form $\hat{G} = G \bigcup_{i=1}^{m} \hat{B}_i$ where
  1. $\hat{B}_1, \hat{B}_2, ..., \hat{B}_m$ are minimally rigid graphs which are either pairwise disjoint or intersect at vertices and edges of G.
  2. $G \bigcap \hat{B}_i = \partial B_i$ for $i = 1, 2, ..., m$.
- (Cruckshank, Kitson and Power)
  Let $\hat{G}$ be a block and hole graph with a single block and finite number of holes. Then

$$\hat{G} \text{ is minimally rigid} \iff \hat{G} \text{ is } (3,6)\text{−tight}$$

References

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- A. Nixon,E. Ross, One brick at a time: a survey of inductive constructions in rigidity theory, Rigidity and symmetry, 22 (2014) , 303-324.
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