Guided waves in pre-stressed hyperelastic plates and tubes

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Mechanical properties of biological tissues

- Useful for diagnostics or for simulations (e.g. stents, head impacts)
- Found from **mechanical tests**, treating tissues as an engineering material.
- **Destructive**, can’t be applied *in vivo*
Mechanical properties of biological tissues

- Alternatively, use non-destructive acoustic waves
- Speed of the wave depends on the material properties
  - Measure wave speed to infer mechanical properties
- Can apply in vivo
Generating waves

- Probe generates a real-time **ultrasound image**
- Also generates a low frequency **shear wave** by focusing acoustic beam
- The wave is seen in the ultrasound field and its **speed is measured**
Elastic Cherenkov effect

Three similar wave phenomena observed when the velocity of the excitation source is greater than the velocities of the resulting waves in the media.
Now we consider **guided waves** (generated using Verasonics device) in a **stretched** polyvinyl alcohol (PVA) cryogel plate immersed in water.
Model wave propagation in solid as small “incremental” deformation superimposed on a large deformation.

Equations of motion:

\[ \nabla \cdot \Sigma = \rho \ddot{u} \]  \hspace{1cm} \text{(solid)}

where \( \Sigma = A_0 \nabla u \) and \( A_0 = \frac{\partial^2 W}{\partial F \partial \dot{F}} \),

\[ \nabla (\kappa \nabla \cdot \mathbf{u}^F) = \rho^F \ddot{\mathbf{u}}^F, \]  \hspace{1cm} \text{(fluid)}

where \( c_p = \sqrt{\kappa / \rho^F} \) is the speed of sound in the fluid.
Dispersion equation

Seeking a wave solution $e^{skx_2}e^{ik(x_1-ct)}$, and imposing continuity of stress and displacements across the fluid-solid interfaces, we find both symmetric and anti-symmetric solutions.

For the anti-symmetric mode, the dispersion equation reads

$$\gamma s_1(1 + s_2^2)^2 \tanh(s_1 kh) - \gamma s_2(1 + s_2^2)^2 \tanh(s_2 kh) + \frac{\rho F c^2}{\sqrt{1 - \frac{c^2}{c_p^2}}} (s_1^2 - s_2^2) = 0.$$

When the plate is not stretched, we recover the equation [3]

$$\left(2 - \frac{\rho c^2}{\mu_0}\right)^2 \tanh(kh_0) - 4 \sqrt{1 - \frac{\rho c^2}{\mu_0}} \tanh \left(\sqrt{1 - \frac{\rho c^2}{\mu_0}} kh_0\right) + \frac{\rho \rho_F c^4}{\mu_0^2 \sqrt{1 - \frac{c^2}{c_p^2}}} = 0,$$

where $\mu_0$ is the shear modulus.
Finite element simulations

Anti-symmetric and symmetric modes
Finite element simulations

- For large radius-to-thickness ratio, FE simulation of waves in a tube agree with theoretical **plate** model.
  - Can use plate theory for tubes (e.g. arteries)
Curve fitting

Determine material parameters by fitting the theoretical curves to the experimental data. For example, the neo-Hookean model was used:

\[ W = \frac{\mu_0}{2} (l_1 - 3). \] (1)

(a) dispersion curves at various stretches, (b) stress response in destructive tensile test

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