Earthquakes as Filippov Systems

Model Analysis

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November 27, 2015

Overview

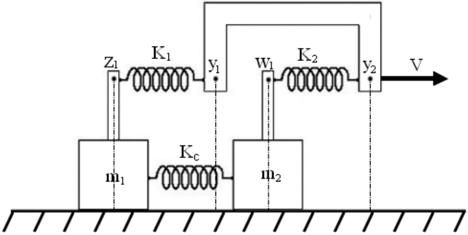
Previous topics:

- Filippov Systems
- Stick-Slip Systems
- Hearthquakes as Stick-Slip phenomenon
- Slider Blocks System
- Mathematical Model
- Simulating

Today topics:

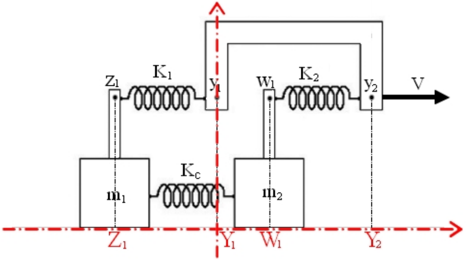
- Find and classify the equilibria
- Apply the Filippov Convex Method

Model: old model



$$\begin{cases} \dot{y}_1 = V \\ \dot{y}_2 = V \\ \dot{z}_1 = z_2 \\ \dot{z}_2 = \frac{1}{m_1}[K_1(y_1 - z_1 - L_1) + K_c(w_1 - z_1 - L_c) - F_1 sgn(z_2)] \\ \dot{w}_1 = w_2 \\ \dot{w}_2 = \frac{1}{m_2}[K_2(y_2 - w_1 - L_2) - K_c(w_1 - z_1 - L_c) - F_2 sgn(w_2)] \end{cases}$$

Model: change of reference



$$\begin{cases} Y_1 = 0 \\ Y_2 = y_2 - y_1 = y \\ Z_1 = z_1 - y_1 \\ Z_2 = z_2 - V \\ W_1 = w_1 - y_1 \\ W_2 = w_2 - V \end{cases} \rightarrow \begin{cases} \dot{Y}_1 = 0 \\ \dot{Y}_2 = 0 \\ \dot{Z}_1 = \dot{z}_1 - V \\ \dot{Z}_2 = \dot{z}_2 \\ \dot{W}_1 = \dot{w}_1 - V \\ \dot{W}_2 = \dot{w}_2 \end{cases}$$

Model: new model

Thanks to the change of reference Y_1 and Y_2 are constant, then they can be considered as parameters.
So the State Space become $\mathbb{R}^4$ and the model is:

$$\begin{cases} \dot{Z}_1 = Z_2 \\ \dot{Z}_2 = \frac{1}{m_1}[K_1(-Z_1 - L_1) + K_c(W_1 - Z_1 - L_c) - F_1 sgn(Z_2 + V)] \\ \dot{W}_1 = W_2 \\ \dot{W}_2 = \frac{1}{m_2}[K_2(y - W_1 - L_2) - K_c(W_1 - Z_1 - L_c) - F_2 sgn(W_2 + V)] \end{cases}$$

In this way we have earned a considerable semplification of the mathematical model at the cost of a harder understanding of the physical behaviour.

Equilibria: without friction

At first let neglet the friction. The model is smooth and linear:

$$\begin{cases} \dot{Z}_1 = Z_2 \\ \dot{Z}_2 = \frac{1}{m_1}[K_1(-Z_1 - L_1) + K_c(W_1 - Z_1 - L_c)] \\ \dot{W}_1 = W_2 \\ \dot{W}_2 = \frac{1}{m_2}[K_2(y - W_1 - L_2) - K_c(W_1 - Z_1 - L_c)] \end{cases}$$

Because of the linearity there's only one equilibrium point given by:

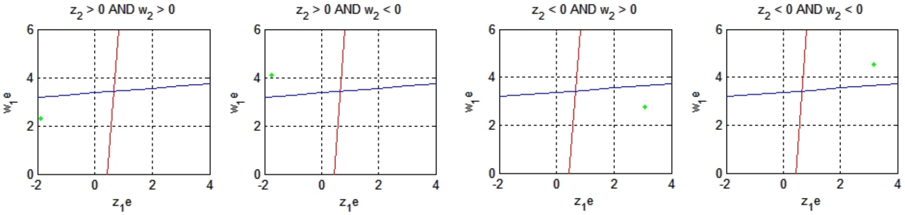
$$\begin{bmatrix} \dot{Z}_1 & \dot{Z}_2 & \dot{W}_1 & \dot{W}_2 \end{bmatrix}^T = \mathbf{0}^T \rightarrow \begin{cases} \dot{Z}_1 = -\frac{(K_2+K_c)(K_1L_1+K_cL_c)+K_c(K_2L_2+K_cLc-K_2y)}{(K_1+K_c)(K_2+K_c)-K_c^2} \\ \dot{Z}_2 = 0 \\ \dot{W}_1 = -\frac{(K_1+K_c)(K_2L_2+K_cL_c-K_2y)+K_c(K_1L_1+K_cL_c)}{(K_1+K_c)(K_2+K_c)-K_c^2} \\ \dot{W}_2 = 0 \end{cases}$$

It's easily possible to demonstrate that the eigenvalues of the jacobian matrix are always pure imaginary then the equilibrium point is always a *center*.

Now let consider the friction again. This divides the state space in 4 regions.

$$1)\ Z_2 > -V \wedge W_2 > -V \qquad 2)\ Z_2 > -V \wedge W_2 < -V$$
$$3)\ Z_2 < -V \wedge W_2 > -V \qquad 4)\ Z_2 < -V \wedge W_2 < -V$$

In each of these regions the coulomb friction is clearly constant. This means that we have just a translation of the equilibrium point according to the sign of the friction.



Filippov Convex Method: overview

Let consider a Filippov system:

$$f(\mathbf{x}) = \begin{cases} \mathrm{f}^{+}(\mathbf{x}) \text{ if } \sigma(\mathbf{x}) > 0 \\ \mathrm{f}^{-}(\mathbf{x}) \text{ if } \sigma(\mathbf{x}) < 0 \end{cases}$$

When a trajectory hits the switching manifold, different behaviours are possible. Which one we are interested now is the *sliding*, that is when it continues to move on this manifold for a while or forever.

The region where this happens is called *sliding surface*: $\Sigma(\mathbf{x}) \subseteq \sigma(\mathbf{x})$

The Filippov Convex Method allows to extend the definition of the vector field on the sliding surface $\Sigma(\mathbf{x})$:

$$f^{\Sigma}(\mathbf{x}) = \frac{f^{+}(\mathbf{x}) + f^{-}(\mathbf{x})}{2} + \frac{f^{-}(\mathbf{x}) - f^{+}(\mathbf{x})}{2}\beta(\mathbf{x})$$

Filippov Convex Method: meaning of $\beta(\mathbf{x})$

The function $\beta(\mathbf{x})$ is defined as:

$$\beta(\mathbf{x}) = -\frac{L_{f^+(\mathbf{x})+f^-(\mathbf{x})}(\sigma(\mathbf{x}))}{L_{f^-(\mathbf{x})-f^+(\mathbf{x})}(\sigma(\mathbf{x}))}$$

where $L_{V(\mathbf{x})}(s(\mathbf{x}))$ is the *Lie Derivative* of the vector field V on the scalar field s:

$$L_{V(\mathbf{x})}(s(\mathbf{x})) = \nabla s(\mathbf{x}) \cdot V(\mathbf{x}) \tag{1}$$

$\beta(\mathbf{x})$ plays a leading role in the Filippov Convex Method. Indeed it's possible to demonstrate that when $-1 < \beta(\mathbf{x}) < 1$ the vector fields from either side of the discontinuity are directed towards one another, so the trajectories are trapped on the switching manifold.
Otherwise if $|\beta(\mathbf{x})| > 1$ the vector fields have the same direction and then the trajectories cross the discontinuity.
So the boundaries of the sliding surface are the region where $\beta(\mathbf{x}) = -1 \wedge \beta(\mathbf{x}) = 1$

Filippov Convex Method: switching manifolds

Let $\mathbf{x} = \begin{bmatrix} Z_1 & Z_2 & W_1 & W_2 \end{bmatrix}$

In the system there are 2 switching manifolds:

$$\begin{aligned} \sigma_z &: z_2 + V = 0 \\ \sigma_w &: w_2 + V = 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} \nabla\sigma_z &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \\ \nabla\sigma_w &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

The Lie derivatives are

$$L_{f_z^+(\mathbf{x})+f_z^-(\mathbf{x})}(\sigma_z(\mathbf{x})) = \frac{1}{m_1}[K_1(-Z_1 - L_1) + K_c(W_1 - Z_1 - L_c)]$$

$$L_{f_z^-(\mathbf{x})-f_z^+(\mathbf{x}}(\sigma_z(\mathbf{x})) = \frac{F_1}{m_1}$$

$$L_{f_w^+(\mathbf{x})+f_w^-(\mathbf{x}}(\sigma_w(\mathbf{x})) = \frac{1}{m_2}[K_2(y - W_1 - L_2) - K_c(W_1 - Z_1 - L_c)]$$

$$L_{f_w^-(\mathbf{x})-f_w^+(\mathbf{x}}(\sigma_w(\mathbf{x})) = \frac{F_2}{m_2}$$

Filippov Convex Method: extended vector fields

That implies:

$$\beta_z = \frac{(K_1 + K_c)Z_1 - K_c W_1 + K_1 L_1 + K_c L_c}{F_1}$$

$$\beta_w = \frac{(K_2 + K_c)W_1 - K_c Z_1 + K_2 L_2 + K_c L_c - K_2 y}{F_1}$$

So the vector fields are:

$$f_z^{\Sigma} = \begin{cases} \dot{Z}_1 = -V \\ \dot{Z}_2 = 0 \\ \dot{W}_1 = W_2 \\ \dot{W}_2 = \frac{1}{m_2}[K_2(y - W_1 - L_2) - K_c(W_1 - Z_1 - L_c) - F_2 sgn(W_2 + V)] \end{cases}$$

$$f_w^{\Sigma} = \begin{cases} \dot{Z}_1 = Z_2 \\ \dot{Z}_2 = \frac{1}{m_1}[K_1(-Z_1 - L_1) + K_c(W_1 - Z_1 - L_c) - F_1 sgn(Z_2 + V)] \\ \dot{W}_1 = -V \\ \dot{W}_2 = 0 \end{cases}$$

Filippov Convex Method: overall sliding vector field

So when $Z_2 = -V \wedge W_2 = -V$ the sliding vector field is:

$$f^{\Sigma} = \begin{cases} \dot{Z}_1 = -V \\ \dot{Z}_2 = 0 \\ \dot{W}_1 = -V \\ \dot{W}_2 = 0 \end{cases}$$

And (using $\beta_z \wedge \beta_w$) the boundaries of the sliding region are:

$$\Sigma_z^- : W_1 = \frac{K_1 + K_c}{K_c} Z_1 + \frac{K_1 L_1 + K_c L_c + F_1}{K_c}$$

$$\Sigma_z^+ : W_1 = \frac{K1 + K_c}{K_c} Z_1 + \frac{K_1 L_1 + K_c L_c - F_1}{K_c}$$

$$\Sigma_z^- : Z_1 = \frac{K_2 + K_c}{K_c} W_1 + \frac{K_2 L_2 + K_c L_c - K_2 y + F_2}{K_c}$$

$$\Sigma_z^+ : Z_1 = \frac{K_2 + K_c}{K_c} + \frac{K_2 L_2 + K_c L_c - K_2 y - F_2}{K_c}$$

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