CONSTRUCTION OF MANIFOLDS WITH NON-NEGATIVE CURVATURE: BIQUOTIENTS

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The Riemannian geometry of an n-dimensional (complete, connected) Riemannian manifold (M^n, g) can be thought of as a study of the deviation of (M^n, g) from being Euclidean. One particular measure of this deviation is the *curvature*. There are several different notions of curvature, but of interest here is the *sectional curvature*. For every point $p \in M^n$ and for every two-dimensional subspace σ of the tangent space T_pM^n at p, the sectional curvature of σ is the Gauss curvature at p of the two-dimensional surface locally determined by the integral curves of tangent vectors in σ.

The restriction of the Euclidean metric on $\mathbb{R}^{n+1}$ to the unit sphere $\mathbf{S}^n$ gives a Riemannian metric on $\mathbf{S}^n$ with constant positive sectional curvature, a so-called *round metric*. As $\mathbf{S}^n$ is, up to scaling and diffeomorphism, the unique simply connected manifold admitting a round metric, it is natural to look for examples of manifolds which satisfy a weaker curvature condition, namely, examples which admit a Riemannian metric of positive or non-negative (sectional) curvature.

For example, the Euclidean metric on $\mathbb{R}^n$ has constant zero curvature, while $\mathbb{R}^n$ also admits a metric of (non-constant) positive curvature. In fact, by work of Gromoll and Meyer [GM1] $\mathbb{R}^n$ is, up to diffeomorphism, the only non-compact manifold which can carry a metric of positive curvature. Furthermore, the Soul Theorem of Cheeger and Gromoll [CG] states that a complete, connected Riemannian manifold (M^n, g) with non-negative curvature contains a compact, totally geodesic submanifold S such that M^n is diffeomorphic to the normal bundle of S. Thus the search for compact examples with positive or non-negative curvature is of particular importance. These notes are concerned with discussing the main source of such examples, namely metrics on biquotients arising from Riemannian submersions from compact Lie groups.

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1. Riemannian Submersions

In Riemannian Geometry there has classicially been much interest in the geometry of submanifolds, corresponding to the induced geometry on the image of immersions $\iota : M^n \to \bar{M}^{n+k}$. A rich theory has been developed, comprising many of the most important concepts in the study of Riemannian manifolds: the Gauss and Codazzi equations; the second fundamental form; totally geodesic and minimal submanifolds.

If one considers instead submersions $\pi : M^{n+k} \to B^n$, a corresponding and similarly important theory can be developed, which turns out to have deep implications in many areas of Riemannian Geometry. Most of the following material can be found in [Be] and [GW].

Definition 1.1. A smooth map $\pi : M^{n+k} \to B^n$ is called a *submersion* if $d\pi_p : T_pM^{n+k} \to T_{\pi(p)}B^n$ is surjective, i.e. $\mathrm{rank}(d\pi_p) = n$, for all $p \in M^{n+k}$.

If $\pi : M^{n+k} \to B^n$ is a submersion and $b \in \pi(M) \subseteq B$, then it follows from the Implicit Function Theorem that the *fibre of π over b*, $F_b := \pi^{-1}(b)$, is a smooth k-dimensional submanifold of M^{n+k}.

Lemma 1.2. *$T_p(F_b) = \ker(d\pi_p)$, for all $p \in F_b$, $b \in B$.*

Proof. For each $v \in T_p(F_b)$ there is a curve $c : (-\varepsilon, \varepsilon) \to F_b$ such that $c(0) = p$ and $c'(0) = v$. Hence $\pi(c(t)) = \pi(p) = b$ for all $t \in (-\varepsilon, \varepsilon)$ and so $0 = \frac{d}{dt}\pi(c(t))|_{t=0} = d\pi_p(v)$. Thus $T_p(F_b) \subseteq \ker(d\pi_p)$ and the claim follows for dimension reasons. $\square$

The *vertical subspace at $p \in M$* is defined by $\mathcal{V}_p := \ker(d\pi_p)$ and the *vertical distribution* by $\mathcal{V} := \{\mathcal{V}_q \mid q \in M\}$. Clearly the restriction of $d\pi_p$ to any complement of $\mathcal{V}_p$ is an isomorphism onto T_bB, where $\pi(p) = b$, although there is no canonical choice for this complementary subspace. However, if M and B are equipped with Riemannian metrics g_M and g_B respectively, the *horizontal subspace at $p \in M$* can be defined as $\mathcal{H}_p := (\mathcal{V}_p)^\perp = \{w \in T_pM \mid g_M(w, \mathcal{V}_p) = 0\}$, and the *horizontal distribution* as $\mathcal{H} := \{\mathcal{H}_q \mid q \in M\}$. Therefore, for all $p \in M$ with $\pi(p) = b \in B$, the tangent space T_pM decomposes as $T_pM = \mathcal{V}_p \oplus \mathcal{H}_p$, and $d\pi_p|_{\mathcal{H}_p} : \mathcal{H}_p \to T_bB$ is an isomorphism. Given a vector field X on M, its vertical and horizontal components will be denoted by $X^{\mathcal{V}} \in \mathcal{V}$ and $X^{\mathcal{H}} \in \mathcal{H}$ respectively. A vector field on M which is π-related to a vector field on B is called *projectable*. A *basic* vector field on M is one that is both projectable and horizontal.

Lemma 1.3. *The space of vertical vector fields on M is an ideal in the algebra of projectable fields on M.*

Proof. If the vector fields X and V on M are projectable and vertical, respectively, then $d\pi([X, V]) = [d\pi(X), 0] = 0$, as desired. $\square$

Definition 1.4. A submersion $\pi : M^{n+k} \to B^n$ between two Riemannian manifolds (M^{n+k}, g_M) and (B^n, g_B) is called a *Riemannian submersion* if

$$g_M(v, w) = g_B(d\pi_p(v), d\pi_p(w)), \text{ for all } p \in M \text{ and } v, w \in \mathcal{H}_p.$$

In order to keep track of the metrics involved, Riemannian submersions will be denoted by $\pi : (M, g_M) \to (B, g_B)$.

Convention. For the sake of simplicity and unless otherwise indicated, all manifolds will be assumed connected and without boundary, all metrics complete and all (Riemannian) submersions surjective.

Lemma 1.5. *Let $\pi : (M, g_M) \to (B, g_B)$ be a Riemannian submersion. Then:*

(a) *If $c : [0,1] \to M$ is a $\mathcal{C}^1$ curve in M, then $L(c) \geqslant L(\pi \circ c)$. Hence, $d_M(p,q) \geqslant d_B(\pi(p), \pi(q))$, for all $p, q \in M$.*

(b) *A smooth vector field X on B has a unique horizontal lift $\widetilde{X}$ on M, i.e. $\widetilde{X}(p) \in \mathcal{H}_p$, $d\pi_p(\widetilde{X}(p)) = X(\pi(p))$, for all $p \in M$.*

(c) *A regular smooth curve c in B has a horizontal lift $\widetilde{c}$ in M, i.e. $\pi(\widetilde{c}(t)) = c(t)$ and $\widetilde{c}'(t) \in \mathcal{H}_{\widetilde{c}(t)}$, for all t. Moreover, $\widetilde{c}$ is unique if $\widetilde{c}(0) \in F_{c(0)}$ is specified.*

(d) *The horizontal lift $\widetilde{\gamma}$ of a geodesic γ in B is a (horizontal) geodesic in M. On the other hand, if σ is a geodesic in M with $\sigma'(0) \in \mathcal{H}_{\sigma(0)}$, then $\sigma'(t) \in \mathcal{H}_{\sigma(t)}$ for all t and $\pi \circ \sigma$ is a geodesic in B.*

(e) *All fibres F_b, $b \in B$, are diffeomorphic. Furthermore, $d_M(p, F_b)$ is constant for all $p \in F_{b'}$, $b, b' \in B$.*

(f) *π is a submetry, i.e. $\pi\left(\overline{\mathcal{B}_r(p)}\right) = \overline{\mathcal{B}_r(\pi(p))}$, for all $r > 0$ and all $p \in M$, where $\overline{\mathcal{B}}_r(x)$ denotes the closure of the ball $\mathcal{B}_r(x)$ of radius r around x.*

(g) *The submersion $\pi : M \to B$ is a locally trivial fibre bundle with fibre $F \cong F_b$, $b \in B$.*

Proof. Recall that $d\pi_{c(t)}|_{\mathcal{H}_{c(t)}}$ is an isometry.

(a) Since $c'(t) = c'(t)^{\mathcal{V}} + c'(t)^{\mathcal{H}}$, it follows that

$$L(\pi \circ c) = \int_0^1 |d\pi_{c(t)}(c'(t))| dt = \int_0^1 |c'(t)^{\mathcal{H}}| dt \leqslant \int_0^1 |c'(t)| dt = L(c).$$

(b) For all $p \in M$ define the vector field $\widetilde{X}$ by

$$\widetilde{X}(p) := (d\pi_p|_{\mathcal{H}_p})^{-1}(X(\pi(p))) \in \mathcal{H}_p.$$

$\widetilde{X}$ is smooth since locally $X(\pi(p)) = \sum a_i(\pi(p)) \frac{\partial}{\partial x_i}$ and, if $\widetilde{\frac{\partial}{\partial x_i}}$ is defined as $(d\pi_p|_{\mathcal{H}_p})^{-1}(\frac{\partial}{\partial x_i})$, then $\widetilde{X}(p) = \sum a_i(\pi(p)) \widetilde{\frac{\partial}{\partial x_i}}$ by linearity.

(c) Let $\widetilde{c}'(t)$ be the horizontal lift of $c'(t) \neq 0$ to M, and let $\widetilde{c}(t)$ be the corresponding integral curves starting at points in $F_{c(0)}$. By the uniqueness of integral curves, $\widetilde{c}(t)$ is unique whenever its initial point $\widetilde{c}(0) \in F_{c(0)}$ is specified. Since M is complete, $\widetilde{c}(t)$ can be continued along the entire domain of definition of c.

(d) Suppose there is some curve $\alpha : [a,b] \to M$ such that $L(\alpha) < L(\widetilde{\gamma}|_{[a,b]})$. Then, by (a) and since $\widetilde{\gamma}'$ is by definition horizontal, $L(\pi \circ \alpha) \leqslant L(\alpha) < L(\widetilde{\gamma}) = L(\gamma)$. This is a contradiction, since γ is locally minimising.

Suppose now that β is a geodesic in B such that $\beta'(0) = d\pi_{\sigma(0)}(\sigma'(0))$, and let $\widetilde{\beta}$ be its horizontal lift starting at $\sigma(0)$. By the above argument, $\widetilde{\beta}$ is a geodesic. However, $\widetilde{\beta}(0) = \sigma(0)$ and $\widetilde{\beta}'(0) = \sigma'(0)$, hence $\widetilde{\beta} = \sigma$, $\sigma'(t)$ is horizontal for all t, and $\pi \circ \sigma = \beta$ is a geodesic in B.

(e) If $c : [0,1] \to B$ is an arbitrary path in B with horizontal lift starting at $p \in F_{c(0)}$ denoted by $\widetilde{c}_p$, then define $\tau_c : F_{c(0)} \to F_{c(1)}$ by $p \mapsto \widetilde{c}_p(1)$. By the uniqueness of horizontal lifts and by considering lifts traversed in the opposite direction, it follows that τ_c is a diffeomorphism.

For $b.b' \in B$, let γ be a geodesic in M minimising the distance from $F_{b'}$ to F_b, and let $p = \gamma(0) \in F_{b'}$. By the first variation formula, γ is orthogonal to both $F_{b'}$ and F_b. From (d), γ must be a horizontal geodesic, hence $L(\pi \circ \gamma) = L(\gamma) = d_M(p, F_b) = d_M(F_{b'}, F_b)$. Therefore every horizontal lift of the geodesic $\pi \circ \gamma$

minimises the distance from $F_{b'}$ to F_b, and

$$d_M(p, F_b) = d_M(q, F_b) = d_M(F_{b'}, F_b), \quad \text{for all } p, q \in F_{b'}.$$

(f) By (a), $\pi\left(\overline{\mathcal{B}_r(p)}\right) \subseteq \overline{\mathcal{B}_r(\pi(p))}$, for all $r > 0$. On the other hand, if γ is a minimising geodesic in B starting at $\pi(p)$ with length at most $r > 0$, the horizontal lift $\widetilde{\gamma}$ in M starting at p has $L(\widetilde{\gamma}) = L(\gamma) \leqslant r$. Thus $\overline{\mathcal{B}_r(\pi(p))} \subseteq \pi\left(\overline{\mathcal{B}_r(p)}\right)$, for all $r > 0$.

(g) Let $b \in B$ and choose $\varepsilon > 0$ such that $\exp_b : \mathcal{B}_\varepsilon(0) \to \mathcal{B}_\varepsilon(b)$ is a diffeomorphism. For a vector $x \in \mathcal{B}_\varepsilon(0) \subseteq T_b B$, let $\widetilde{x} : F_b \to \mathcal{H}$ denote the (smoothly varying and unique) horizontal lifts of x along F_b, i.e. $d\pi_p(\widetilde{x}(p)) = x$ for all $p \in F_b$. Consider the smooth map

$$\varphi : F_b \times \mathcal{B}_\varepsilon(0) \to \pi^{-1}(B_\varepsilon(b)) = B_\varepsilon(F_b)$$

defined by $\varphi(p, x) = \exp_p(\widetilde{x}(p))$. If $q \in \mathcal{B}_\varepsilon(F_b)$ then, by the choice of ε, there is a unique minimal geodesic $\gamma : [0,1] \to B$ from $\pi(q)$ to b. The horizontal lift $\widetilde{\gamma}_q$ of γ with $\widetilde{\gamma}_q(0) = q$ has $p := \widetilde{\gamma}_q(1) \in F_b$. Hence $q = \varphi(p, -\widetilde{\gamma}_q'(1))$, i.e. φ is surjective. On the other hand, φ is injective by the uniqueness of $\widetilde{\gamma}_q$. Moreover, $d\varphi(p, x)$ has maximal rank since $\exp_p$ is a local diffeomorphism. Therefore, the map

$$(id_{F_b} \times \exp_b) \circ h^{-1} : \pi^{-1}(\mathcal{B}_\varepsilon(b)) \to F_b \times \mathcal{B}_\varepsilon(0) \to F_b \times \mathcal{B}_\varepsilon(b)$$

is a diffeomorphism. Since $b \in B$ was arbitrary and all fibres are diffeomorphic, π is a locally trivial fibre bundle.

□

Notation. Suppose $\pi : (M, g_M) \to (B, g_B)$ is a Riemannian submersion. The following notational conventions will be used throughout these notes, often without further elaboration:

- If X is a smooth vector field on B, then $\widetilde{X}$ denotes its unique horizontal lift to M.
- Vector fields on B will be denoted by X, Y, Z or W.
- Vertical vector fields on M will always be denoted by U or V.
- Respectively, ∇^N, R^N and $\sec_N$ denote the Levi-Civita connection, Riemannian curvature tensor and sectional curvature of a Riemannian manifold $(N, g_N) = (N, \langle\, ,\, \rangle_N)$, where

$$R^N_{X,Y} Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_X \nabla^N_Y Z - \nabla^N_{[X,Y]} Z \quad \text{and}$$
$$\sec_N(X, Y) = \frac{\langle R^N_{X,Y} Y, X\rangle_N}{|X|^2 |Y|^2 - \langle X, Y\rangle_N^2}.$$

Proposition 1.6. *Let G be a compact Lie group acting freely and isometrically on a Riemannian manifold (M, g_M). Then the quotient map $\pi : M \to M/G$ induces a Riemannian metric $\breve{g}$ on M/G such that $\pi : (M, g_M) \to (M/G, \breve{g})$ is a Riemannian submersion.*

Proof. Let $p \in M$ and $x \in T_{\pi(p)}(M/G)$. Since $d\pi_p|_{\mathcal{H}_p}$ is an isomorphism, there is a unique vector $\widetilde{x}_p \in \mathcal{H}_p$ such that $d\pi_p(\widetilde{x}_p) = x$. Define

$$|x| := |\widetilde{x}_p| = \sqrt{g_M(\widetilde{x}_p, \widetilde{x}_p)}.$$

This is a well-defined norm on the tangent vectors to M/G. Indeed, if $q \in \pi^{-1}(\pi(p)) = G \cdot p$, then there is $g \in G$ such that $q = g \cdot p$ and a unique vector $\widetilde{x}_q \in \mathcal{H}_q$ such that

$d\pi_q(\widetilde{x}_q) = x$. Since G preserves orbits and hence horizontal spaces, $dg_p : \mathcal{H}_p \to \mathcal{H}_q$. Furthermore, $\pi \circ g = \pi$. Therefore

$$d\pi_q(\widetilde{x}_q) = x = d\pi_p(\widetilde{x}_p) = d(\pi \circ g)_p(\widetilde{x}_p) = d\pi_q(dg_p(\widetilde{x}_p)).$$

As $d\pi_q|_{\mathcal{H}_q}$ is an isomorphism, it follows that $\widetilde{x}_q = dg_p(\widetilde{x}_p)$. Since G acts by isometries, $|\widetilde{x}_q| = |\widetilde{x}_p|$.

The norm $|\cdot|$ can now be extended to a Riemannian metric $\check{g}$ on M/G by polarisation, where smoothness of $\check{g}$ follows from the smoothness of the metric g_M, the quotient map π and the action of G. □

Example 1.7. The Lie group $\mathbf{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ acts freely and isometrically on the standard round 3-dimensional sphere of radius 1, $\mathbf{S}^3(1) = \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}$, via the so-called Hopf action

$$z \cdot (u,v) = (zu, zv), \quad z \in \mathbf{S}^1, (u,v) \in \mathbf{S}^3.$$

The quotient of this action is a round 2-dimensional sphere of radius $\frac{1}{2}$, $\mathbf{S}^2(\frac{1}{2})$. Indeed, if $(u,v) \in \mathbf{S}^3$ with $u \neq 0$, then there is a unique point of the form $(t,w) \in \mathbf{S}^3$, $t \in \mathbb{R}$, $t > 0$, in the orbit of (u,v). The set of all such points forms an open hemi-sphere Σ in the unit 2-sphere in $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$, the boundary of which is the $\mathbf{S}^1$ orbit of $(0,1)$ (i.e. $u = 0$ above).

The quotient map $\pi : \mathbf{S}^3 \to \mathbf{S}^3/\mathbf{S}^1$ identifies the points on the boundary of this hemi-sphere to a point, and all other orbits to a point in the interior of the hemi-sphere Σ. Therefore, $\mathbf{S}^3/\mathbf{S}^1$ is a 2-dimensional sphere. The length of a great circle in this quotient is π, as this is the length of a geodesic between antipodal boundary points of Σ and passing through the point $(1,0) \in \Sigma$. Thus the quotient 2-sphere has radius $\frac{1}{2}$ as desired.

It is also clear that if the $\mathbf{S}^1$ orbits are scaled so that they shrink (uniformly) to points, then one gets convergence (in the sense of Gromov-Hausdorff) of a sequence of (Berger) metrics on $\mathbf{S}^3$ to the round $\mathbf{S}^2(\frac{1}{2})$, seen via the shrinking of the boundary of Σ to a point.

Theorem 1.8. *Let $\pi : (M, g_M) \to (B, g_B)$ be a Riemannian submersion and let X, Y, Z and W denote smooth vector fields on B. Then:*

(a) $\nabla^M_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla^B_X Y} + \frac{1}{2}[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}$. *In particular,* $d\pi(\nabla^M_{\widetilde{X}} \widetilde{Y}) = \nabla^B_X Y$.

(b) *For all $p \in M$, $(\nabla^M_{\widetilde{X}} \widetilde{Y})^{\mathcal{V}}(p) = \frac{1}{2}[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}(p)$ depends only on $\widetilde{X}(p), \widetilde{Y}(p) \in \mathcal{H}_p$.*

(c) *The curvature tensor R^B of (B, g_B) is given by*

$$\begin{aligned}\langle R^B_{X,Y} Z, W\rangle_B =& \langle R^M_{\widetilde{X},\widetilde{Y}} \widetilde{Z}, \widetilde{W}\rangle_M + \tfrac{1}{4}\langle [\widetilde{Y}, \widetilde{Z}]^{\mathcal{V}}, [\widetilde{X}, \widetilde{W}]^{\mathcal{V}}\rangle_M \\ &- \tfrac{1}{4}\langle [\widetilde{X}, \widetilde{Z}]^{\mathcal{V}}, [\widetilde{Y}, \widetilde{W}]^{\mathcal{V}}\rangle_M - \tfrac{1}{2}\langle [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, [\widetilde{Z}, \widetilde{W}]^{\mathcal{V}}\rangle_M.\end{aligned}$$

(d) *If X and Y are, in addition, orthonormal, then*

$$\sec_B(X,Y) = \sec_M(\widetilde{X}, \widetilde{Y}) + \frac{3}{4}\left|[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}\right|^2. \tag{1.1}$$

Proof. Let $V \in \mathcal{V}$ be a vertical vector field on M.

(a) Clearly $\langle \widetilde{X}, V\rangle_M = \langle \widetilde{Y}, V\rangle_M = \langle \widetilde{Z}, V\rangle_M = 0$ and $\widetilde{X}\langle \widetilde{Y}, \widetilde{Z}\rangle_M = X\langle Y, Z\rangle_B$. By Lemma 1.3, $d\pi([\widetilde{X}, V]) = 0$, whereas $[X,Y] = [d\pi(\widetilde{X}), d\pi(\widetilde{Y})] = d\pi([\widetilde{X}, \widetilde{Y}])$. Hence $\langle [\widetilde{X}, V], \widetilde{Y}\rangle_M = 0$ and $\langle [\widetilde{X}, \widetilde{Y}], \widetilde{Z}\rangle_M = \langle [X,Y], Z\rangle_B$. Moreover, as $\langle \widetilde{X}, \widetilde{Y}\rangle_M$ is constant along the fibres, $V\langle \widetilde{X}, \widetilde{Y}\rangle_M = 0$. Therefore, by the Koszul formula,

$$\langle \nabla^M_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}\rangle_M = \langle \nabla^B_X Y, Z\rangle_B \quad \text{and} \quad \langle \nabla^M_{\widetilde{X}} \widetilde{Y}, V\rangle_M = \tfrac{1}{2}\langle [\widetilde{X}, \widetilde{Y}], V\rangle_M,$$

which yields $\nabla^M_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla^B_X Y} + \frac{1}{2}[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}$ as desired.

(b) Since ∇^M is the Levi-Civita connection of (M, g_M),

$$\begin{aligned}\langle [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, V\rangle_M &= \langle [\widetilde{X}, \widetilde{Y}], V\rangle_M \\ &= \langle \nabla^M_{\widetilde{X}} \widetilde{Y} - \nabla^M_{\widetilde{Y}} \widetilde{X}, V\rangle_M \\ &= \langle \widetilde{X}, \nabla^M_{\widetilde{Y}} V\rangle_M - \langle \widetilde{Y}, \nabla^M_{\widetilde{X}} V\rangle_M,\end{aligned}$$

from which the claim easily follows.

(c) By (a), $\widetilde{X}\langle \nabla^M_{\widetilde{Y}} \widetilde{Z}, \widetilde{W}\rangle_M = X\langle \nabla^B_Y Z, W\rangle_B$ and, since V and $[V, \widetilde{X}]$ are vertical,

$$\langle \nabla^M_V \widetilde{X}, \widetilde{Y}\rangle_M = \langle \nabla^M_{\widetilde{X}} V, \widetilde{Y}\rangle_M = -\langle V, \nabla^M_{\widetilde{X}} \widetilde{Y}\rangle_M = -\langle V, \tfrac{1}{2}[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}\rangle_M.$$

Therefore

$$\begin{aligned}\langle \nabla^M_{\widetilde{X}} \nabla^M_{\widetilde{Y}} \widetilde{Z}, \widetilde{W}\rangle_M &= \widetilde{X}\langle \nabla^M_{\widetilde{Y}} \widetilde{Z}, \widetilde{W}\rangle_M - \langle \nabla^M_{\widetilde{Y}} \widetilde{Z}, \nabla^M_{\widetilde{X}} \widetilde{W}\rangle_M \\ &= X\langle \nabla^B_Y Z, W\rangle_B - \langle \nabla^M_{\widetilde{Y}} \widetilde{Z}, \nabla^M_{\widetilde{X}} \widetilde{W}\rangle_M \\ &= \langle \nabla^B_X \nabla^B_Y Z, W\rangle_B + \langle \nabla^B_Y Z, \nabla^B_X W\rangle_B \\ &\quad - \langle \widetilde{\nabla^B_Y Z} + \tfrac{1}{2}[\widetilde{Y}, \widetilde{Z}]^{\mathcal{V}}, \widetilde{\nabla^B_X W} + \tfrac{1}{2}[\widetilde{X}, \widetilde{W}]^{\mathcal{V}}\rangle_M \\ &= \langle \nabla^B_X \nabla^B_Y Z, W\rangle_B - \tfrac{1}{4}\langle [\widetilde{Y}, \widetilde{Z}]^{\mathcal{V}}, [\widetilde{X}, \widetilde{W}]^{\mathcal{V}}\rangle_M,\end{aligned} \tag{1.2}$$

while, on the other hand,

$$\begin{aligned}\langle \nabla^M_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}, \widetilde{W}\rangle_M &= \langle \nabla^M_{[\widetilde{X}, \widetilde{Y}]^{\mathcal{H}}} \widetilde{Z}, \widetilde{W}\rangle_M + \langle \nabla^M_{[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}} \widetilde{Z}, \widetilde{W}\rangle_M \\ &= \langle \nabla^B_{[X,Y]} Z, W\rangle_B - \tfrac{1}{2}\langle [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, [\widetilde{Z}, \widetilde{W}]^{\mathcal{V}}\rangle_M.\end{aligned} \tag{1.3}$$

Combining equations (1.2) and (1.3) with the definition of R^M yields the desired result.

(d) This follows directly from (c) by setting $Z = Y$, $W = X$.

□

Remark 1.9. Theorem 1.8 shows, in particular, that Riemannian submersions are sectional curvature non-decreasing. Hence, if (M, g_M) has non-negative sectional curvature, so too does (B, g_B), and one can further hope that the extra term $\frac{3}{4}|[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}|^2$ in fact ensures that (B, g_B) is positively curved.

Notice further that a plane tangent to B is the image of a horizontal plane tangent to M. Thus, in order to determine if (B, g_B) has any zero-curvature planes, one need only examine horizontal planes tangent to M.

Definition 1.10. Let $\pi : (M, g_M) \to (B, g_B)$ be a Riemannian submersion.

(a) The *A-tensor* (or *O'Neill tensor*) is the tensor field $A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ on M defined by

$$A_{\widetilde{X}} \widetilde{Y} = (\nabla^M_{\widetilde{X}} \widetilde{Y})^{\mathcal{V}} = \tfrac{1}{2}[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}},$$

where $\widetilde{X}$ and $\widetilde{Y}$ are horizontal vector fields.

(b) The *T-tensor* is the tensor field $T : \mathcal{V} \times \mathcal{V} \to \mathcal{H}$ on M given by

$$T_U V = (\nabla^M_U V)^{\mathcal{H}},$$

where U and V are vertical vector fields.

Remark 1.11. The A-tensor vanishes if and only if the horizontal distribution $\mathcal{H}$ is integrable, in which case (M, g_M) is locally isometric to $(B \times F_b, g_B \oplus g_{F_b})$. The T-tensor, on the other hand, is the second fundamental form of the fibres and vanishes if and only if the fibres are totally geodesic, in which case all fibres are isometric. If both $A \equiv 0$ and $T \equiv 0$, then (M, g_M) is locally isometric to $(B \times F, g_B \oplus g_F)$, i.e. the metric on the fibre is independent of the base point.

2. Bi-invariant metrics on Lie groups

It is well known (and a standard exercise to show) that a compact Lie group G can be equipped with a *bi-invariant metric*, i.e. a metric $\langle\, , \rangle_0$ for which the *left-* and *right-multiplication* maps, $L_g : G \to G;\ h \mapsto gh$ and $R_g : G \to G;\ h \mapsto hg$, are isometries for all $g \in G$. More generally, a metric $\langle\, , \rangle$ on G is called *left-invariant* (resp. *right-invariant*) if L_g (resp. R_g) is an isometry for all $g \in G$. Given any inner product on the Lie algebra $\mathfrak{g}$ of G, one can construct a left- (resp. right-) invariant metric on G via propagation of the inner product by left- (resp. right-) multiplication.

It turns out that almost every known example of a manifold with a metric of non-negative or positive sectional curvature is constructed via a Riemannian submersion from a compact Lie group G equipped with a bi-invariant metric $\langle\, , \rangle_0$. The goal of this section is to show that $(G, \langle\, , \rangle_0)$ is non-negatively curved, from which it follows immediately via (1.1) that the base of any Riemannian submersion $\pi : (G, \langle\, , \rangle_0) \to (B, g_B)$ is also non-negatively curved.

Notation. From now on, unless explicitly stated otherwise:

- G will always denote a compact Lie group and $\mathfrak{g}$ its Lie algebra.
- The map $L_g : G \to G$ denotes left-multiplication by $g \in G$, i.e. $L_g(h) = gh$. Similarly, $R_g : G \to G$ denotes right-multiplication by $g \in G$, i.e. $R_g(h) = hg$.
- If $v \in \mathfrak{g}$, then X_v denotes the corresponding *left-invariant vector field*, i.e. $X_v(g) = (dL_g)_e(v)$, where e denotes the identity element of G.
- The *exponential map* of G, $\exp : \mathfrak{g} \to G$, is defined by $\exp(v) = \varphi_v(1, e)$, where $\varphi_v : \mathbb{R} \times G \to G$ denotes the flow of the corresponding left-invariant vector field X_v. In particular, $exp((s + t)v) = exp(sv) \exp(tv)$, for all $s, t \in \mathbb{R}$.
- The *adjoint representation* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is given by
$$\mathrm{Ad}_g := (dL_g)_{g^{-1}} \circ (dR_{g^{-1}})_e = (dR_{g^{-1}})_g \circ (dL_g)_e,$$
namely, the derivative at $e \in G$ of the conjugation map $h \mapsto ghg^{-1}$.
- A bi-invariant metric on G is denoted by $\langle\, , \rangle_0$ or g_0, and its corresponding Levi-Civita connection, Riemannian curvature tensor and sectional curvature by ∇^0, R^0 and $\sec_0$, respectively.
- If K is a closed subgroup of G, its Lie algebra is given by $\mathfrak{k}$ and there is an Ad_K-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to $\langle\, , \rangle_0$.

Lemma 2.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

(a) *If X and Y are left-invariant vector fields on G, then the Lie bracket $[X, Y]$ is also a left-invariant vector field. Hence $(dL_g)_h([X, Y](h)) = [(dL_g)_h X, (dL_g)_h Y](gh)$*
(b) *The flow φ_v of the left-invariant vector field X_v corresponding to $v \in \mathfrak{g}$ is given by $\varphi_v(t, \cdot) = R_{\exp(tv)}$.*
(c) *For all $v, w \in \mathfrak{g}$, $\frac{d}{dt} \mathrm{Ad}_{\exp(tv)}\, w \,\big|_{t=0} = [v, w]$.*
(d) *If $\langle\, , \rangle_0$ is a bi-invariant metric on G, then*
$$\langle [u, v], w\rangle_0 = -\langle v, [u, w]\rangle_0, \ \textit{for all } u, v, w \in \mathfrak{g}. \tag{2.1}$$

Proof. (a) Let $f : G \to \mathbb{R}$ be an arbitrary smooth function. Notice that, for all $g, h \in G$,

$$\begin{aligned}((Yf) \circ L_g)(h) &= Y(gh)f \\ &= ((dL_g)_h Y(h))f \\ &= Y(h)(f \circ L_g) \\ &= (Y(f \circ L_g))(h).\end{aligned}$$

Consequently, left-invariance of $[X, Y]$ follows from

$$\begin{aligned}[X,Y](g)f &= X(g)(Yf) - Y(g)(Xf) \\ &= ((dL_g)_e X(e))(Yf) - ((dL_g)_e Y(e))(Xf) \\ &= X(e)((Yf) \circ L_g) - Y(e)((Xf) \circ L_g) \\ &= X(e)(Y(f \circ L_g)) - Y(e)(Y(f \circ L_g)) \\ &= [X,Y](e)(f \circ L_g) \\ &= ((dL_g)_e [X,Y](e))f.\end{aligned}$$

(b) For all $g \in G$, the curve $\varphi_v(t, g) := g \cdot \exp(tv) = R_{\exp(tv)} g$ clearly has derivative $(dL_g)_e v = X_v(g)$, as desired.

(c) Recall that, if X and Y are two vector fields on a manifold M, then

$$[X,Y](p) = (\mathcal{L}_X Y)(p) = \tfrac{d}{dt}(d\varphi(-t, \cdot))_{\varphi(t,p)}(Y(\varphi(t,p))) \big|_{t=0},$$

where $\mathcal{L}$ is the Lie derivative and $\varphi(t, p)$ is the flow of X through $p \in M$. Thus,

$$\begin{aligned}\tfrac{d}{dt} \mathrm{Ad}_{\exp(tv)} w \big|_{t=0} &= \tfrac{d}{dt}(dR_{\exp(-tv)})_{\exp(tv)}(dL_{\exp(tv)})_e w \big|_{t=0} \\ &= \tfrac{d}{dt}(dR_{\exp(-tv)})_{\exp(tv)}(X_w(\exp(tv)) \big|_{t=0} \\ &= [X_v, X_w](e) \\ &= [v, w].\end{aligned}$$

(d) Since $\langle w, w\rangle_0 = \langle \mathrm{Ad}_g w, \mathrm{Ad}_g w\rangle_0$, for all $g \in G$, $w \in \mathfrak{g}$, it follows that

$$\begin{aligned}0 &= \tfrac{1}{2}\tfrac{d}{dt}\langle \mathrm{Ad}_{\exp(tv)} w, \mathrm{Ad}_{\exp(tv)} w\rangle_0\big|_{t=0} \\ &= \langle \tfrac{d}{dt} \mathrm{Ad}_{\exp(tv)} w \big|_{t=0}, w\rangle_0 \\ &= \langle [v, w], w\rangle_0, \quad \text{by (c)},\end{aligned}$$

from which one easily deduces equation (2.1).

□

Theorem 2.2. *Let G be a compact Lie group equipped with a bi-invariant metric $\langle\, ,\, \rangle_0$. If X, Y and Z are left-invariant vector fields on G, then:*

(a) $\nabla^0_X Y = \frac{1}{2}[X, Y]$.

(b) $R^0_{X,Y} Z = -\frac{1}{4}[[X, Y], Z]$.

(c) *If X and Y are orthonormal, then* $\sec_0(X, Y) = \frac{1}{4}|[X, Y]|^2$.

Proof. From the Koszul formula, together with Lemma 2.1 and the bi-invariance of the metric, one easily derives $\nabla^0_X X = 0$, for all left-invariant vector fields X on G. Part (a) then follows immediately from the symmetry of the connection ∇^0.

Given (a) and (d) of Lemma 2.1, parts (b) and (c) follow directly from the definitions.

□

Remark 2.3. In particular, Theorem 2.2 shows that a compact Lie group equipped with a bi-invariant metric is always non-negatively curved. On the other hand, one can easily show (see [Wa]) that the only Lie groups admitting a bi-invariant (or, more generally, left-invariant) metric with positive sectional curvature are $\mathrm{SO}(3)$ and $\mathbf{S}^3 = \mathrm{Sp}(1) = \mathrm{SU}(2)$.

3. Cheeger deformations

Suppose that a Riemannian manifold (M, g_M) is non-negatively curved and a compact Lie group K acts on (M, g_M) by isometries, i.e. $K \subseteq \mathrm{Isom}(M, g_M)$. Then one can perform a so-called *Cheeger deformation* (see [Ch]) on the metric g_M to obtain a new K-invariant metric $\check{g}_M$ which also has non-negative curvature, but potentially has less zero-curvature planes. Indeed, consider the product manifold $(M\times K, g_M \oplus g_0)$, where g_0 is a bi-invariant metric on K. The Lie group K acts on $(M \times K, g_M \oplus g_0)$ by isometries via

$$g \star (p, k') = (k \cdot p, kk'),$$

where $k \in K$, $(p, k') \in M \times K$, and $\cdot$ denotes the action of K on M. Moreover, the action $\star$ on the product is free, since $k \star (p, k') = (p, k')$ if and only if $k = e$. Hence the quotient $M \times_K K := (M \times K)/K$ is a manifold. In fact, $M \times_K K$ is diffeomorphic to M, where the diffeomorphism is induced by the map $M \times K \to M;\ (p, k) \mapsto k^{-1} \cdot p$. By Proposition 1.6, there is an induced metric $\check{g}_M$ on M such that the quotient map

$$\pi : (M \times K, g_M \oplus g_0) \to (M \times_K K, \check{g}_M) = (M, \check{g}_M)$$

is a Riemannian submersion.

In particular, if (M, g_M) is non-negatively curved, then so too is $(M, \check{g}_M)$. Finally, notice that K acts on $(M, \check{g}_M)$ by isometries, i.e. via the action induced by the isometric action of K on the right-hand side of (K, g_0).

Suppose now that $(M, g_M) = (G, g_0)$ is a compact Lie group equipped with a bi-invariant metric, and let K be a closed subgroup of G. By the procedure above one can easily equip G with a family of left-invariant metrics which are only right K-invariant. Indeed, the required Cheeger deformation is given by equipping K with the bi-invariant metric $tg_0|_K$, where $t > 0$, and yields a family g_λ, $\lambda \in (0, 1)$, of left-invariant, right-K-invariant metrics on G.

More explicitly, if the action $\cdot$ of K on G is given by $k \cdot g = gk^{-1}$, $k \in K$, $g \in G$, then the action $\star$ of K on $G \times K$ is given by $k \star (g, k') = (gk^{-1}, kk')$, $k \in K$, $(g, k') \in G \times K$, and the diffeomorphism $G \times_K K \to G$ is induced by the map $G \times K \to G;\ (g, k) \to gk$.

In this situation, it is easy to compute the vertical and horizontal subspaces, and to determine the metric g_λ explicitly: Let $\mathfrak{k}$ denote the Lie algebra of K and $\mathfrak{p} \subseteq \mathfrak{g}$ its orthogonal complement with respect to g_0. Hence the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is Ad_k-invariant, for all $k \in K$. For $v \in \mathfrak{g}$ and $w \in \mathfrak{k}$, consider the curve

$$\alpha(t) := \pi(g \exp(tv), k \exp(tw)) = g \exp(tv) k \exp(tw)$$

in G with $\alpha(0) = gk$. Clearly

$$\begin{aligned} d\pi_(g, k)((dL_g)_e v, (dL_k)_e w) &= \alpha'(0) \\ &= (dL_g)_k (dR_k)_e v + (dL_{gk})_e w \\ &= (dL_{gk})_e (\mathrm{Ad}_{k^{-1}} v + w). \end{aligned}$$

Since the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\mathrm{Ad}_{k^{-1}}$-invariant,

$$\begin{aligned}((dL_g)_e v, (dL_k)_e w) \in \ker(d\pi_(g,k)) \subseteq T_{(g,k)}(G \times K) \\ \iff v_{\mathfrak{p}} = 0 \text{ and } w = -\operatorname{Ad}_{k^{-1}} v_{\mathfrak{k}},\end{aligned}$$

where $v_{\mathfrak{p}}$ and $v_{\mathfrak{k}}$ denote the $\mathfrak{p}$ and $\mathfrak{k}$ components of $v \in \mathfrak{g}$ respectively. Therefore (for dimension reasons) the vertical subspace at (g,k) is given by

$$\mathcal{V}_{(g,k)} = \{((dL_g)_e v, -(dL_k)_e \operatorname{Ad}_{k^{-1}} v) \mid v \in \mathfrak{k}\}.$$

It is a simple exercise to verify that the horizontal subspace at (g,k) with respect to the metric $g_0 \oplus tg_0$ is of the form

$$\mathcal{H}_{(g,k)} = \{((dL_g)_e x, \tfrac{1}{t}(dL_k)_e \operatorname{Ad}_{k^{-1}} x_{\mathfrak{k}}) \mid x \in \mathfrak{g}\}, \tag{3.1}$$

from which it follows that the isomorphism $d\pi_{(g,k)}|_{\mathcal{H}_{(g,k)}}$ is given by

$$\begin{aligned}d\pi_{(g,k)}((dL_g)_e x, \tfrac{1}{t}(dL_k)_e \operatorname{Ad}_{k^{-1}} x_{\mathfrak{k}}) &= (dL_{gk})_e \operatorname{Ad}_{k^{-1}}(x + \tfrac{1}{t}x_{\mathfrak{k}}) \\ &= (dL_{gk})_e \operatorname{Ad}_{k^{-1}}(x_{\mathfrak{p}} + \tfrac{t+1}{t}x_{\mathfrak{k}}).\end{aligned} \tag{3.2}$$

Consider now the left-invariant vector field X_w on G defined via $X_w(g) := (dL_g)_e w$. From (3.2),the horizontal lift $\widetilde{X}_w$ at a point $(gk^{-1}, k) \in \pi^{-1}(g)$ of X_w can be written as

$$\begin{aligned}\widetilde{X}_w(gk^{-1}, k) &= (d\pi_{(g,k)})^{-1}(X_w(g)) \\ &= ((dL_{gk^{-1}})_e \operatorname{Ad}_k(w_{\mathfrak{p}} + \tfrac{t}{t+1}w_{\mathfrak{k}}), \tfrac{1}{t+1}(dL_k)_e w_{\mathfrak{k}}).\end{aligned} \tag{3.3}$$

Finally, setting $\lambda := \frac{t}{t+1} \in (0,1)$ and using the bi-invariance of g_0, it follows that the metric g_λ induced on G by the Riemannian submersion π is determined by

$$\begin{aligned}g_\lambda(X_w(g), X_v(g)) &= (g_0 \oplus tg_0)(\widetilde{X}_w(gk^{-1}, k), \widetilde{X}_v(gk^{-1}, k)) \\ &= g_0(w_{\mathfrak{p}} + \tfrac{t}{t+1}w_{\mathfrak{k}}, v_{\mathfrak{p}} + \tfrac{t}{t+1}v_{\mathfrak{k}}) + tg_0(\tfrac{1}{t+1}w_{\mathfrak{k}}, \tfrac{1}{t+1}v_{\mathfrak{k}}) \\ &= g_0(w_{\mathfrak{p}}, v_{\mathfrak{p}}) + \lambda g_0(w_{\mathfrak{k}}, v_{\mathfrak{k}}).\end{aligned}$$

In particular, this yields an explicit verification of the left-invariance of the metric g_λ. Moreover, since the product metric $g_0 \oplus tg_0$ on $G \times K$ has non-negative sectional curvature, so too does (G, g_λ). In many situations, g_λ has fewer zero-curvature planes than the bi-invariant metric g_0.

Lemma 3.1 ([Es1]). *With the notation as above, if $K \subseteq G$ is a symmetric pair, i.e. $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, then $v, w \in \mathfrak{g}$ (hence $X_v(g), X_w(g) \in T_gG$) span a zero-curvature plane with respect to g_λ if and only if*

$$[v, w] = [v_{\mathfrak{k}}, w_{\mathfrak{k}}] = [v_{\mathfrak{p}}, w_{\mathfrak{p}}] = 0.$$

Proof. By the invariance of the metrics, it suffices to restrict attention to horizontal planes at $(e,e) \in G \times K$.

By (1.1), the horizontal lift of a g_λ-zero-curvature plane will have zero curvature with respect to the bi-invariant metric $g_0 \oplus tg_0$, i.e. for $v, w \in \mathfrak{g}$, $\sec_{g_\lambda}(v, w) = 0$ implies $\sec_0(\widetilde{X}_v(e,e), \widetilde{X}_w(e,e)) = 0$. This second identity is true if and only if

$$\begin{aligned}0 &= \left[\left(v_{\mathfrak{p}} + \tfrac{t}{t+1}v_{\mathfrak{k}}, \tfrac{1}{t+1}v_{\mathfrak{k}}\right), \left(w_{\mathfrak{p}} + \tfrac{t}{t+1}w_{\mathfrak{k}}, \tfrac{1}{t+1}w_{\mathfrak{k}}\right)\right] \\ &= \left(\left[v_{\mathfrak{p}} + \tfrac{t}{t+1}v_{\mathfrak{k}}, w_{\mathfrak{p}} + \tfrac{t}{t+1}w_{\mathfrak{k}}\right], \left[\tfrac{1}{t+1}v_{\mathfrak{k}}, \tfrac{1}{t+1}w_{\mathfrak{k}}\right]\right) \\ &= \left([v_{\mathfrak{p}}, w_{\mathfrak{p}}] + \left(\tfrac{t}{t+1}\right)^2 [v_{\mathfrak{k}}, w_{\mathfrak{k}}] + \tfrac{t}{t+1}\left([\, v_{\mathfrak{k}}, w_{\mathfrak{p}}] + [v_{\mathfrak{p}}, w_{\mathfrak{k}}t]\right), \left(\tfrac{t}{t+1}\right)^2 [v_{\mathfrak{k}}, w_{\mathfrak{k}}]\right).\end{aligned}$$

This is equivalent to $[v_{\mathfrak{k}}, w_{\mathfrak{k}}] = 0$ and $[v_{\mathfrak{p}}, w_{\mathfrak{p}}] + \frac{t}{t+1}([\,v_{\mathfrak{k}}, w_{\mathfrak{p}}] + [v_{\mathfrak{p}}, w_{\mathfrak{k}}]) = 0$. Since $K \subseteq G$ is a symmetric pair and since bi-invariance of g_0 implies $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$, the second identity yields $[v_{\mathfrak{p}}, w_{\mathfrak{p}}] = 0$ and $[\,v_{\mathfrak{k}}, w_{\mathfrak{p}}] + [v_{\mathfrak{p}}, w_{\mathfrak{k}}] = 0$, from which the assertion follows.

Conversely, if $\sec_0(\widetilde{X}_v(e,e), \widetilde{X}_w(e,e)) = 0$, then one can compute directly, or else apply a recent result of Tapp [Ta], to show that the corresponding plane spanned by $v, w \in \mathfrak{g}$ must also have zero curvature. □

Remark 3.2. Note that the result of Tapp [Ta] mentioned above states that, given any compact Lie group G equipped with a bi-invariant metric g_0 and a Riemannian submersion $\pi : (G, g_0) \to (B, g_B)$, the A-tensor will never improve a horizontal g_0-zero-curvature plane on G to a positively-curved plane on (B, g_B).

4. Biquotients

In his Habilitation [Es1], Eschenburg initiated the systematic study of biquotients, following the interesting example observed by Gromoll and Meyer in [GM2]. Most of the material in this section is taken from [Es1].

Let G be a compact Lie group, $U \subset G \times G$ a closed subgroup, and let U act on G via

$$(u_1, u_2) \cdot g = u_1 g u_2^{-1}, \;\; g \in G, (u_1, u_2) \in U. \tag{4.1}$$

The action (4.1) is *free*, hence the quotient $G/\!\!/U$ is a manifold called a *biquotient*, if and only if, whenever $(u_1, u_2) \cdot g = g$ for some $g \in G$, then $(u_1, u_2) = (e, e)$. This is equivalent to the condition that, for all non-trivial $(u_1, u_2) \in U$, u_1 is never conjugate to u_2 in G. If, on the other hand, the action is not free, the quotient $G/\!\!/U$ will still be a manifold so long as U acts *effectively freely*, i.e. any element $(u_1, u_2) \in U$ which fixes some $g \in G$ fixes all of G, that is, (u_1, u_2) lies in the *ineffective kernel*. For an action of the form (4.1), the ineffective kernel is given by $U \cap \Delta Z_G$, where ΔZ_G denotes the diagonal embedding of the centre Z_G of G in $G \times G$. This follows since, if $(u_1, u_2) \in U$ fixes some $g \in G$, then (u_1, u_2) fixes all $g \in G$, in particular $e \in G$. Hence $u_1 = u_2 \in Z_G$.

As a special case, a *homogeneous space* G/H is a biquotient $G/\!\!/U$ with $U = \{e\} \times H \subseteq G \times G$. The action (4.1) of $U = \{e\} \times H$ on G is clearly always free. Furthermore, if a closed subgroup $K \subseteq G$ acts (effectively) freely on the left of a homogeneous space G/H, then the quotient $K\backslash G/H$ is a biquotient with $U = K \times H$. As it turns out, all biquotients can be written in this form. Indeed, as first noticed by Eschenburg [Es2], before being used geometrically by Wilking [Wi], the map $G \times G \to G$; $(g_1, g_2) \mapsto g_1^{-1} g_2$ induces a diffeomorphism

$$\Delta G \backslash (G \times G)/U \to G/\!\!/U,$$

where ΔG denotes the diagonal embedding of G in $G \times G$ and $\Delta G \times U$ acts on $G \times G$ via

$$((g, g), (u_1, u_2)) \cdot (g_1, g_2) = (g g_1 u_1^{-1}, g g_2 u_2^{-1}).$$

By Theorems 1.8 and 2.2, it is clear that if G is equipped with a bi-invariant metric, then any biquotient $G/\!\!/U$ inherits a metric with non-negative sectional curvature. More generally, if $K \subseteq G$ is a closed subgroup, $U \subseteq G \times K \subseteq G \times G$ and G is equipped with a left-invariant, right-K-invariant metric $\langle\, , \rangle$ (for example, via a Cheeger deformation), then U acts by isometries on G and therefore the submersion $G \to G/\!\!/U$ induces a metric on $G/\!\!/U$ from the metric on G, such that the quotient map is a Riemannian submersion. This is encoded in the notation $(G, \langle\, , \rangle)/\!\!/U$.

For $g \in G$, consider the groups

$$
\begin{aligned}
U_L^g &:= \{(gu_1g^{-1}, u_2) \mid (u_1, u_2) \in U\}, \\
U_R^g &:= \{(u_1, gu_2g^{-1}) \mid (u_1, u_2) \in U\}, \quad \text{and} \\
\widehat{U} &:= \{(u_2, u_1) \mid (u_1, u_2) \in U\}.
\end{aligned} \tag{4.2}
$$

It's a trivial exercise to check that U_L^g, U_R^g and $\widehat{U}$ act (effectively) freely via (4.1) on G whenever U does.

Lemma 4.1. *Using the notation above, if G is equipped with a left-invariant, right-K-invariant metric $\langle\,,\,\rangle$, then $(G, \langle\,,\,\rangle)/\!\!/U$ is isometric to $(G, \langle\,,\,\rangle)/\!\!/U_L^g$, diffeomorphic to $G/\!\!/U_R^g$ (isometric if $g \in K$), and diffeomorphic to $G/\!\!/\widehat{U}$ (isometric if and only if $\langle\,,\,\rangle$ is bi-invariant).*

Proof. In the case of U_L^g this follows from the fact that left-translation $L_g : G \to G$ is an isometry which satisfies $gu_1g^{-1}(L_g g')u_2^{-1} = L_g(u_1 g' u_2^{-1})$. Therefore L_g induces an isometry of the orbit spaces $G/\!\!/U$ and $G/\!\!/U_L^g$. Similarly, $R_{g^{-1}}$ induces a diffeomorphism between $G/\!\!/U$ and $G/\!\!/U_R^g$, which is an isometry if $g \in K$.

Finally, the actions of U and $\widehat{U}$ are equivariant under the diffeomorphism $\tau : G \to G$, $\tau(g) := g^{-1}$. That is, $u_1\tau(g)u_2^{-1} = \tau(u_2 g u_1^{-1})$. Notice that this is an isometry if and only if $\langle\,,\,\rangle$ is bi-invariant, since for $v \in \mathfrak{g}$ the identities

$$
d\tau_g((dL_g)_e v) = -(dR_{g^{-1}})_e v = -(dL_{g^{-1}})_e \operatorname{Ad}_g v,
$$

together with left-invariance of $\langle\,,\,\rangle$, imply that τ is an isometry if and only if $|v| = |d\tau_g((dL_g)_e v)| = |\operatorname{Ad}_g v|$, for all $g \in G$ and $v \in \mathfrak{g}$. Therefore, $G/\!\!/U$ and $G/\!\!/\widehat{U}$ are, in general, diffeomorphic but not isometric. $\square$

Example 4.2. The first example of a biquotient was given by Gromoll and Meyer in [GM2], where they observed that Milnor's 7-dimensional exotic sphere $\Sigma^7_{2,-1}$ could be written as a biquotient $\mathrm{Sp}(2)/\!\!/\mathrm{Sp}(1)$, where $\mathrm{Sp}(1)$ acts on $\mathrm{Sp}(2)$ via

$$
q \cdot A = \begin{pmatrix} q & \\ & q \end{pmatrix} A \begin{pmatrix} \bar{q} & \\ & 1 \end{pmatrix}, \quad q \in \mathrm{Sp}(1), A \in \mathrm{Sp}(2).
$$

Consequently, $\Sigma^7_{2,-1}$ admits a metric with non-negative sectional curvature, the first such example among the exotic spheres. Totaro [To] and Kapovitch and Ziller [KZ] have subsequently shown that $\Sigma^7_{2,-1}$ is the only exotic sphere of any dimension that can be written as a biquotient. Nevertheless, Grove and Ziller [GZ] have constructed metrics with non-negative curvature on several other Milnor spheres via a different method.

It is now natural to ask which groups $U \subseteq G \times G$ can act (effectively) freely on G via the action (4.1). The first thing to be aware of is that, for a given group U, one need only check for freeness on elements of the maximal torus of U. Indeed, if S is a maximal torus of U then, without loss of generality, it may be assumed that $S \subseteq T \times T$, where T is a maximal torus of G (see the proof of Lemma 4.3 below). As every element $(u_1, u_2) \in U$ is conjugate via some $(h_1, h_2) \in U$ to an element $(s_1, s_2) \in S$, one has $u_1 g u_2^{-1} = g$, for some $g \in G$, if and only if $h_1 s_1 h_1^{-1} = u_1 = g u_2 g^{-1} = g h_2 s_2 h_2^{-1} g^{-1}$, i.e. if and only if $s_1 = (h_1^{-1} g h_2) s_2 (h_1^{-1} g h_2)^{-1}$. In other words, if and only if there is some element w of the Weyl group $W_G := N_G(T)/T$ of $T \subseteq G$ such that $s_1 = w s_2 w^{-1}$ as elements of T.

Lemma 4.3. *Let G be a compact Lie group and let $S \subseteq G \times G$ be a torus acting effectively freely on G via the action* (4.1)*. If T is a maximal torus of G, then*

$$\dim(S/(S \cap \Delta Z_G)) \leqslant \dim(T).$$

Consequently, if a closed subgroup $U \subseteq G \times G$ acts effectively freely on G via (4.1)*, then*

$$\mathrm{rank}(U/(U \cap \Delta Z_G) \leqslant \mathrm{rank}(G).$$

Proof. As $S \subseteq G \times G$ is a torus, it must be contained in some $T' \times T \subseteq G \times G$, where T' and T are maximal tori of G. Since all maximal tori of G are conjugate, there is some $g \in G$ such that $T' = g^{-1}Tg$. By the discussion surrounding the groups given in (4.2), $S_L^g := \{(gs_1g^{-1}, s_2) \mid (s_1, s_2) \in S\} \subseteq T \times T$ also acts effectively freely on G. Therefore, without loss of generality, assume $S \subseteq T \times T$ for some maximal torus T of G.

Notice that $S \cap \Delta T := \{(s,s) \in S \mid s \in T\} = S \cap \Delta Z_G$. Indeed, if $(s,s) \in S \cap \Delta T$, then $sts^{-1} = t$ for all $t \in T \in G$. However, since the action is effectively free, this implies that $(s,s) \in \Delta Z_G$. On the other hand, if $(s,s) \in S \cap \Delta Z_G$, then $sgs^{-1} = g$ for all $g \in G$, and in particular for $g \in T$. Hence $s \in T$, since T is a maximal torus. Thus $S \cap \Delta Z_G \subseteq \Delta T$.

Finally, the desired inequality follows since

$$S/(S \cap \Delta Z_G) = S/(S \cap \Delta T) \subseteq (T \times T)/\Delta T \cong T.$$

□

In the search for examples of biquotients with positive sectional curvature, the possible choices of U are even more restrictive.

Theorem 4.4. *Let G be a compact Lie group equipped with a left-invariant, right-K-invariant metric $\langle\, ,\, \rangle$ and let $U \subseteq G \times K \subseteq G \times G$ be a closed subgroup acting on $(G, \langle\, ,\, \rangle)$ effectively freely and isometrically via the action* (4.1)*. Suppose further that $U \cap (K \times K)$ contains a maximal torus of U. Then:*

(a) *There is a compact Lie group H equipped with a left-invariant metric $\langle\!\langle\, ,\, \rangle\!\rangle$ and a totally geodesic isometric immersion $f : (H, \langle\!\langle\, ,\, \rangle\!\rangle) \to (G, \langle\, ,\, \rangle)/\!\!/U$ such that*

$$\mathrm{rank}(H) = \mathrm{rank}(G) - \mathrm{rank}(U/(U \cap \Delta Z_G).$$

(b) *If $(G, \langle\, ,\, \rangle)/\!\!/U$ has positive sectional curvature, then*

$$\mathrm{rank}(G) - \mathrm{rank}(U/(U \cap \Delta Z_G)) = \begin{cases} 0 & \textit{for } \dim(G/\!\!/U) \textit{ even,} \\ 1 & \textit{for } \dim(G/\!\!/U) \textit{ odd.} \end{cases}$$

Proof. (a) Let S be a maximal torus of U contained in $U \cap (K \times K)$. As in the proof of Lemma 4.3, it may be assumed without loss of generality that $S \subseteq T' \times T'$, where T' is a maximal torus of K. Extend T' to a maximal torus T of G, such that $S \subseteq T' \times T' \subseteq T \times T$.

Let $C := C_G(T')$ be the centraliser of T' in G. Note that $\mathrm{rank}(C) = \mathrm{rank}(G)$, since C contains the maximal torus T. Moreover, C is a totally geodesic Lie subgroup of G, since it is fixed by all isometries of $(G, \langle\, ,\, \rangle)$ given by conjugation by an element of $T' \subseteq K$.

By hypothesis, $S \subseteq T' \times T'$, hence the biquotient $C/\!\!/S$ is, in fact, a homogeneous space C/S', where $S' := \{s_1 s_2^{-1} \mid (s_1, s_2) \in S\} \subseteq T' \subseteq C$. However, S' is, by definition, in the centre of C. Therefore $H := G/\!\!/S = C/S'$ is a compact Lie group. Moreover, the kernel of the homomorphism $\varphi : S \to S';\ (s_1, s_2) \mapsto s_1 s_2^{-1}$

consists of elements of the form $(s,s) \in S$. However, since S acts on G effectively freely, such elements must lie in ΔZ_G. Therefore $S' \cong S/(S \cap \Delta Z_G)$, from which it follows that $\dim(S') = \dim(S/(S \cap \Delta Z_G)) = \mathrm{rank}(U/(U \cap \Delta Z_G))$. Thus $H = C/S'$ has $\mathrm{rank}(H) = \mathrm{rank}(G) - \mathrm{rank}(U/(U \cap \Delta Z_G))$.

It remains to show that there is a totally geodesic, isometric immersion of H into $(G, \langle\, ,\, \rangle)/\!\!/U$. Let $\mathfrak{u} := \mathfrak{u}_1 \oplus \mathfrak{u}_2 \subseteq \mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{s} := \mathfrak{s}_1 \oplus \mathfrak{s}_2 \subseteq \mathfrak{u}$ denote the Lie algebras of U and $S \subseteq U$ respectively. By general Lie theory, $\mathfrak{u} = \mathfrak{s} \oplus \bigoplus_\lambda E(\lambda)$, where the linear maps $\lambda : \mathfrak{s} \to \mathbb{R}$ run through all roots of U, and $E(\lambda)$ denotes the (2-dimensional) root space corresponding to the root λ. Moreover, for every $x \in E(\lambda) \subseteq \mathfrak{u}$, there is a $y \in E(\lambda)$ such that $[v,x] = -\lambda(v)y$ and $[v,y] = \lambda(v)x$, for all $v \in \mathfrak{s}$. Call this condition $(*)$.

The inclusion $C \hookrightarrow G$ induces a map $f : H = C/\!\!/S \to G/\!\!/U$. This will be a totally geodesic, isometric immersion, if f maps the S-horizontal distribution in C into the U-horizontal distribution in G. As the metric $\langle\, ,\, \rangle$ on G is left-invariant and $H = C/S'$ is homogeneous, this will be the case if, for all $c \in C$, $\langle v, w \rangle = 0$ for all $v \in \mathfrak{m}$ and $w \in (dL_{c^{-1}})_c \mathcal{V}_c \subseteq \mathfrak{g}$, where

$$(dL_{c^{-1}})_c \mathcal{V}_c = \{\mathrm{Ad}_{c^{-1}} w_1 - w_2 \mid (w_1, w_2) \in \mathfrak{u}\}$$

is the U-vertical subspace at $c \in C \subseteq G$ translated back to $e \in G$, and $\mathfrak{m} \subseteq \mathfrak{g}$ is the $\langle\, ,\, \rangle$-orthogonal complement of the Lie algebra $\mathfrak{s}'$ of S' in the Lie algebra $\mathfrak{c}$ of C, i.e. $\mathfrak{c} = \mathfrak{s}' \oplus \mathfrak{m} \subseteq \mathfrak{g}$. To achieve this, consider the decomposition $\mathfrak{u} = \mathfrak{s} \oplus \bigoplus_\lambda E(\lambda)$.

First, note that any vector $v = (v_1, v_2) \in \mathfrak{s}$ maps to $v_1 - v_2 \in \mathfrak{s}'$ under the surjection $d\varphi_{(e,e)}$. Hence, by definition, $\mathfrak{m}$ is orthogonal to $v_1 - v_2 = \mathrm{Ad}_{c^{-1}} v_1 - v_2 \in (dL_{c^{-1}})_c \mathcal{V}_c$, for all $(v_1, v_2) \in \mathfrak{s}$ and for all $c \in C$, where $v_1 = \mathrm{Ad}_{c^{-1}} v_1$ because $S \subseteq T' \times T'$ and $\mathrm{Ad}_{c^{-1}}$ acts trivially on the Lie algebra of T'.

On the other hand, given a root λ of U, choose $v = (v_1, v_2) \in \mathfrak{s}$ such that $\lambda(v) = 1$. For $x = (x_1, x_2) \in E(\lambda) \subseteq \mathfrak{u}$, choose $y = (y_1, y_2) \in E(\lambda)$ such that condition $(*)$ is satisfied. Then $x_2 = [v_2, y_2]$. On the other hand, since C commutes with T' and $S \subseteq T' \times T'$, it follows that $[v_2, z] = 0$ for all $z \in \mathfrak{c}$. Hence, using the skew-symmetry induced by the $T' \subseteq K$-invariance of the metric,

$$\langle x_2, z \rangle = \langle [v_2, y_2], z \rangle = -\langle y_2, [v_2, z] \rangle = 0, \quad \text{for all } z \in \mathfrak{c}.$$

Similarly, as Ad_c acts trivially on the Lie algebra of T',

$$\langle z, \mathrm{Ad}_c x_1 \rangle = \langle z, \mathrm{Ad}_c [v_1, y_1] \rangle = \langle z, [v_1, \mathrm{Ad}_c y_1] \rangle = \langle [z, v_1], \mathrm{Ad}_c y_1 \rangle = 0,$$

for all $z \in \mathfrak{c}$. Therefore, for all $c \in C$, $\mathfrak{c}$ is orthogonal to all vectors $\mathrm{Ad}_{c^{-1}} w_1 - w_2 \in (dL_{c^{-1}})_c \mathcal{V}_c \subseteq \mathfrak{g}$, where $(w_1, w_2) \in \bigoplus_\lambda E(\lambda)$. In particular, $\mathfrak{m} \subseteq \mathfrak{c}$ is orthogonal to such vectors. In combination with the previous paragraph, this argument yields the assertion.

(b) Given f, a totally geodesic, isometric immersion as in (a), it is clear that either $\dim(H) = \dim(f(H)) \leqslant 1$ or $(H, \langle\!\langle\, ,\, \rangle\!\rangle)$ is positively curved. Hence, as demonstrated by Wallach in [Wa], H must be one of $\{e\}$, $\mathbf{S}^1$, $\mathrm{SO}(3)$ or $\mathbf{S}^3$, i.e. H must have rank at most 1. Since $\dim(G/\!\!/U) = \dim(G) - \dim(U/(U \cap \Delta Z_G)$ and, furthermore, every compact Lie group L satisfies $\dim(L) = \mathrm{rank}(L) + 2d_L$, where d_L is the number of root spaces of L, the claim follows from (a).

□

5. HOMOGENEOUS EXAMPLES WITH POSITIVE SECTIONAL CURVATURE

Although there are certainly non-simply connected examples of manifolds with positive sectional curvature, in this section only simply connected manifolds will be considered. This case is already the most interesting, since there are no known obstructions to prevent a non-negatively curved, simply connected manifold from admitting a metric with positive curvature.

The basic example of a positively curved manifold is the sphere $\mathbf{S}^n \subseteq \mathbb{R}^{n+1}$ equipped with the standard round metric g_{rd} inherited from the Euclidean metric on $\mathbb{R}^{n+1}$. In the special cases of $\mathbf{S}^{2n+1} \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2(n+1)}$ and $\mathbf{S}^{4n+3} \subseteq \mathbb{H}^{n+1} = \mathbb{R}^{4(n+1)}$, there are free, isometric actions of $\mathbf{S}^1 \subseteq \mathbb{C}$ and $\mathbf{S}^3 \subseteq \mathbb{H}$ respectively given by

$$x \cdot (y_1, \dots, y_{n+1}) = (xy_1, \dots, xy_{n+1}),$$

where $(x, (y_1, \dots, y_{n+1})) \in \mathbf{S}^1 \times \mathbf{S}^{2n+1}$ or $\mathbf{S}^3 \times \mathbf{S}^{4n+3}$ accordingly. These are the so called *Hopf actions* and the quotients are $\mathbf{CP}^n$ and $\mathbf{HP}^n$ respectively. By Proposition 1.6 and Theorem 1.8, $\mathbf{CP}^n$ and $\mathbf{HP}^n$ therefore admit metrics with positive sectional curvature.

Notice that the spheres $\mathbf{S}^n \subseteq \mathbb{R}^{n+1}$, $\mathbf{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ and $\mathbf{S}^{4n+3} \subseteq \mathbb{H}^{n+1}$ can be identified with the first columns of elements of the compact, simple Lie groups $\mathrm{SO}(n+1)$, $\mathrm{SU}(n+1)$ and $\mathrm{Sp}(n+1)$ respectively. This identification is, in fact, encoded in the description of these spheres as homogeneous spaces as follows:

$$\mathbf{S}^n = \mathrm{SO}(n+1)/\,\mathrm{SO}(n),\ \mathbf{S}^{2n+1} = \mathrm{SU}(n+1)/\,\mathrm{SU}(n)\ \text{ and } \mathbf{S}^{4n+3} = \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n).$$

Moreover, a bi-invariant metric on $\mathrm{SO}(n+1)$, $\mathrm{SU}(n+1)$ and $\mathrm{Sp}(n+1)$ induces positive curvature on the corresponding sphere. However, it should be noted that the induced metrics on $\mathbf{S}^{2n+1}$ and $\mathbf{S}^{4n+3}$ are not the round metrics. In each case, the round metric can be achieved by performing a suitable Cheeger deformation of the bi-invariant metric. Nevertheless, one can use the description of the spheres as *normal* homogeneous spaces (i.e. homogeneous spaces $(G, \langle\, , \rangle_0)/H$, where $\langle\, , \rangle_0$ is a bi-invariant metric) to construct Riemannian submersions onto $\mathbf{CP}^n$ and $\mathbf{HP}^n$, where the induced metrics are the same as those discussed above coming from the round spheres.

Suppose K is a closed subgroup of a compact Lie group G. Consider the K-principal bundle $K \to G \to G/K$. For any closed subgroup $H \subseteq K \subseteq G$, there is an associated bundle

$$K/H \to G \times_K (K/H) \to G/K, \tag{5.1}$$

where the $G \times_K (K/H) = (G \times (K/H))/K$, where the action of K on the product $G \times (K/H)$ is given by $k \cdot (g, k'H) = (gk^{-1}, kk'H)$. It is clear that this action of K commutes with the action of H on the right of K, hence

$$G \times_K (K/H) = (G \times_K K)/H \cong G/H.$$

Therefore, the associated bundle (5.1) corresponding to a triple $H \subseteq K \subseteq G$ can be rewritten as

$$K/H \to G/H \to G/K$$

and is called a *homogeneous fibration*. In particular, equipping G with a bi-invariant metric turns the projection $(G, \langle\, , \rangle_0)/H \to (G, \langle\, , \rangle_0)/K$ into a Riemannian submersion.

Returning to the description of the spheres $\mathbf{S}^{2n+1}$ and $\mathbf{S}^{4n+3}$ as normal homogeneous spaces, note that there is in each case an intermediate subgroup, namely

$$\mathrm{SU}(n) \subseteq \mathrm{U}(n) \subseteq \mathrm{SU}(n+1) \quad \text{and} \quad \mathrm{Sp}(n) \subseteq \mathrm{Sp}(n) \times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(n+1).$$

In this way one achieves the Riemannian submersions alluded to above:

$$\mathrm{U}(n)/\mathrm{SU}(n) \to (\mathrm{SU}(n+1), \langle\,,\,\rangle_0)/\mathrm{SU}(n) \to (\mathrm{SU}(n+1), \langle\,,\,\rangle_0)/\mathrm{U}(n),$$

where $\mathbf{S}^1 = \mathrm{U}(n)/\mathrm{SU}(n)$ and $\mathbf{CP}^n = \mathrm{SU}(n+1)/\mathrm{U}(n)$, and

$$(\mathrm{Sp}(n)\times\mathrm{Sp}(1))/\mathrm{Sp}(n) \to (\mathrm{Sp}(n+1), \langle\,,\,\rangle_0)/\mathrm{Sp}(n) \to (\mathrm{Sp}(n+1), \langle\,,\,\rangle_0)/(\mathrm{Sp}(n)\times\mathrm{Sp}(1)),$$

where $\mathbf{S}^3 = (\mathrm{Sp}(n)\times\mathrm{Sp}(1))/\mathrm{Sp}(n)$ and $\mathbf{HP}^n = \mathrm{Sp}(n+1)/(\mathrm{Sp}(n)\times\mathrm{Sp}(1))$.

Notice that the spaces

$$\begin{aligned}(\mathbf{S}^n, g_{rd}) &= (\mathrm{SO}(n+1), \langle\,,\,\rangle_0)/\mathrm{SO}(n),\\ (\mathbf{CP}^n, g) &= (\mathrm{SU}(n+1), \langle\,,\,\rangle_0)/\mathrm{U}(n), \text{ and}\\ (\mathbf{HP}^n, g) &= (\mathrm{Sp}(n+1), \langle\,,\,\rangle_0)/(\mathrm{Sp}(n)\times\mathrm{Sp}(1))\end{aligned}$$

are compact rank 1 symmetric spaces (called *CROSSes*), where $(G, \langle\,,\,\rangle_0)/K$ is symmetric of rank 1 if every vector in the orthogonal complement $\mathfrak{p}$ of the Lie algebra $\mathfrak{k}$ of K in $\mathfrak{g}$ commutes only with multiples of itself (i.e. $[x,y]=0$, $x,y\in\mathfrak{p}$, implies $y=\lambda x$, some $\lambda\in\mathbb{R}$). By Theorem 1.8, such spaces admit positive curvature. There is only one other CROSS, namely the *Cayley projective plane*

$$(\mathbf{CaP}^2, g) = (\mathrm{F}_4, \langle\,,\,\rangle_0)/\mathrm{Spin}(9),$$

although the details in this case are beyond the scope of these notes.

In fact, almost all homogeneous examples admitting positive curvature can be constructed using homogeneous fibrations. Notice first, by comparing with the construction in Section 3, that in a homogeneous fibration (5.1) a metric g_λ, $\lambda\in(0,1)$, on G achieved via a Cheeger deformation of a bi-invariant metric $\langle\,,\,\rangle_0$ in the direction of the subgroup K will still induce a Riemannian submersion $(G,g_\lambda)/H \to (G,g_\lambda)/K$, and that the metric on the base will be isometric to the metric induced by the bi-invariant metric on G (since vectors orthogonal to K are unaffected by the Cheeger deformation). Furthermore, the metric on the fibre K/H is simply a scaling by $\lambda\in(0,1)$ of the metric induced by $\langle\,,\,\rangle_0$.

The metric on G/H can be written down explicitly. Indeed, if $\mathfrak{h}\subseteq\mathfrak{k}\subseteq\mathfrak{g}$ denote the Lie algebras of $H\subseteq K\subseteq G$, then one has orthogonal decompositions

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p} \quad\text{and}\quad \mathfrak{k} = \mathfrak{h}\oplus\mathfrak{m}$$

with respect to the bi-invariant metric $\langle\,,\,\rangle_0$. In particular, $\mathfrak{p}$ is isomorphic to the tangent space $T_{[K]}(G/K)$, $\mathfrak{m}$ to $T_{[H]}(K/H)$, and $\mathfrak{m}\oplus\mathfrak{p}$ to $T_{[H]}(G/H)$. Then the metric g_λ on G induces a metric on G/H which is given by

$$\langle x,y\rangle_\lambda := \langle x_\mathfrak{p}, y_\mathfrak{p}\rangle_0 + \lambda\langle x_\mathfrak{m}, y_\mathfrak{m}\rangle_0, \quad \text{for } x,y\in\mathfrak{m}\oplus\mathfrak{p}. \tag{5.2}$$

Theorem 5.1 ([Wa]). *Given a homogeneous fibration $K/H\to G/H\to G/K$ associated to a triple $H\subseteq K\subseteq G$ of compact Lie groups and with the same notation as above, assume that:*

(a) *The base $(G,\langle\,,\,\rangle_0)/K$ is a compact rank 1 symmetric space.*
(b) *The fibre $(K,\langle\,,\,\rangle_0)/H$ is positively curved.*
(c) *If $x\in\mathfrak{p}$ and $y\in\mathfrak{m}$ are both non-zero, then $[x,y]\neq 0$.*

Then $(G,g_\lambda)/H = (G/H,\langle\,,\,\rangle_\lambda)$, $\lambda\in(0,1)$, has positive sectional curvature.

Proof. As $(G,g_\lambda)\to(G/H,\langle\,,\,\rangle_\lambda)$ is a Riemannian submersion, if the plane spanned by $x,y\in\mathfrak{m}\oplus\mathfrak{p}\cong T_{[H]}(G/H)$ has zero curvature with respect to $\langle\,,\,\rangle_\lambda$, then it also has zero curvature with respect to g_λ on G. By the Eschenburg Lemma 3.1 and since $\mathfrak{m} \in \mathfrak{k}$, this is true if and only if

$$0 = [x, y] = [x_\mathfrak{p}, y_\mathfrak{p}] = [x_\mathfrak{m}, y_\mathfrak{m}].$$

By (a), $x_\mathfrak{p}$ and $y_\mathfrak{p}$ must be linearly dependent, so without loss of generality $x_\mathfrak{p} = 0$. On the other hand, by (b), $x_\mathfrak{m}$ and $y_\mathfrak{m}$ must be linearly dependent, hence without loss of generality $y_\mathfrak{m} = 0$. Therefore $x \in \mathfrak{m}$ and $y \in \mathfrak{p}$ with $[x, y] = 0$, contradicting (c), so there cannot be any zero-curvature planes. □

Positively curved homogeneous spaces were classified in the 70's by Wallach [Wa] and Bérard-Bergery [BB]. Besides the CROSSes mentioned before, examples occur only in dimensions 6, 7, 12, 13 and 24. Ignoring metrics for the moment, the entire list consists of the following manifolds (cf. [Zi2]):

(a) The (Wallach) flag manifolds:

$$\begin{aligned} W^6 &= \mathrm{SU}(3)/T^2 \\ W^{12} &= \mathrm{Sp}(3)/\,\mathrm{Sp}(1)^3 \\ W^{24} &= \mathrm{F}_4\,/\,\mathrm{Spin}(8) \end{aligned}$$

(b) The Berger spaces:

$$\begin{aligned} B^7 &= \mathrm{SO}(5)/\,\mathrm{SO}(3)_{\max} \\ B^{13} &= \mathrm{SU}(5)/(\mathrm{Sp}(2)\cdot \mathbf{S}^1) \end{aligned}$$

(c) The Aloff-Wallach spaces [AW]:

$$W^7_{k,l} = \mathrm{SU}(3)/\mathbf{S}^1_{k,l}, \quad \text{for } \gcd(k,l) = 1,\ kl(k+l) \neq 0.$$

For the Berger space B^7, the maximal subgroup $\mathrm{SO}(3)_{\max} \subseteq \mathrm{SO}(5)$ is given by the unique five-dimensional representation determined by the action of $\mathrm{SO}(3)$ (via conjugation) on

$$\mathbb{R}^5 = \{A \in M_{3\times 3}(\mathbb{R}) \mid A^t = A,\ \mathrm{tr}(A) = 0\}.$$

It turns out that B^7 is isotropy irreducible, hence there is a unique homogeneous metric up to scaling. In particular, $(B^7, g) = (\mathrm{SO}(5), \langle\,,\,\rangle_0)/\,\mathrm{SO}(3)_{\max}$ is positively curved, as can be seen from a direct computation which will not appear here.

In all other cases Theorem 5.1 can be applied to establish positive curvature:

(a) From the inclusions $T^2 \subseteq \mathrm{U}(2) \subseteq \mathrm{SU}(3)$, $\mathrm{Sp}(1)^3 \subseteq \mathrm{Sp}(2)\times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(3)$ and $\mathrm{Spin}(8) \subseteq \mathrm{Spin}(9) \subseteq \mathrm{F}_4$ one obtains the homogeneous fibrations

$$\begin{aligned} &\mathbf{S}^2 \to W^6 \to \mathbf{CP}^2 \\ &\mathbf{S}^4 \to W^{12} \to \mathbf{HP}^2 \\ &\mathbf{S}^8 \to W^{24} \to \mathbf{CaP}^2. \end{aligned}$$

As all three cases are similar, positive curvature will be demonstrated only for W^6. It is clear that both $(\mathrm{U}(2), \langle\,,\,\rangle_0)/T^2 = (\mathbf{S}^2, \hat{g})$ and $(\mathrm{SU}(3), \langle\,,\,\rangle_0)/\,\mathrm{U}(2) = (\mathbf{CP}^2, \check{g})$ are CROSSes, hence the assumptions (a) and (b) hold in Theorem 5.1. In this case, given $T^2 \subseteq \mathrm{U}(2) \subseteq \mathrm{SU}(3)$, one has

$$\begin{aligned}
\mathfrak{h} &= \{\mathrm{diag}(is, it, -i(s+t)) \mid s, t \in R\}, \\
\mathfrak{m} &= \left\{\mathrm{diag}(V, 0) \mid V = \left(\begin{smallmatrix} 0 & v \\ -\bar{v} & 0 \end{smallmatrix} \right),\ v \in \mathbb{C} \right\}, \\
\mathfrak{p} &= \left\{ \left(\begin{smallmatrix} 0 & 0 & w_1 \\ 0 & 0 & w_2 \\ -\bar{w}_1 & -\bar{w}_2 & 0 \end{smallmatrix} \right) \mid w = (w_1, w_2)^t \in \mathbb{C}^2 \right\}.
\end{aligned}$$

It is now a trivial exercise to check that $[x, y] = 0$ for $x \in \mathfrak{p}$, $y \in \mathfrak{m}$, if and only if either $x = 0$ or $y = 0$. Consequently, $(W^6, \langle\, ,\, \rangle_\lambda) = (\mathrm{SU}(3), g_\lambda)/T^2$ has positive curvature by Theorem 5.1.

(b) For the Berger space B^{13}, the subgroup $\mathrm{Sp}(2) \cdot \mathbf{S}^1 \subseteq \mathrm{SU}(5)$ consists of matrices of the form $\mathrm{diag}(\bar{z}^4, zA)$, where $z \in \mathbf{S}^1 \subseteq \mathbb{C}$, $A \in \mathrm{Sp}(2)$, and $\mathrm{Sp}(2) \subseteq \mathrm{SU}(4)$ via the standard embedding. Therefore,

$$\mathrm{Sp}(2) \cdot \mathbf{S}^1 = (\mathrm{Sp}(2) \times \mathbf{S}^1)/\{\pm(\mathrm{Id}, 1)\} \subseteq \mathrm{U}(4) \subseteq \mathrm{SU}(5).$$

Then B^{13} occurs as the total space of a fibration

$$\mathbf{RP}^5 \to B^{13} \to \mathbf{CP}^4,$$

where the fibre is $\mathrm{U}(4)/(\mathrm{Sp}(2) \cdot \mathbf{S}^1) = \mathrm{SU}(4)/(\mathrm{Sp}(2) \cdot \mathbb{Z}_2) = \mathrm{SO}(6)/\mathrm{O}(5) = \mathbf{RP}^5$. A bi-invariant metric on $\mathrm{SU}(5)$ induces a bi-invariant metric on $\mathrm{SO}(6)$, which in turn induces a metric with constant positive curvature on the fibre $\mathbf{RP}^5$. As the base $(\mathrm{SU}(5), \langle\, ,\, \rangle_0)/\mathrm{U}(4) = (\mathbf{CP}^4, \check{g})$ is a CROSS, it follows from Theorem 5.1 that $(B^{13}, \langle\, ,\, \rangle_\lambda) = (\mathrm{SU}(5), g_\lambda)/(\mathrm{Sp}(2) \cdot \mathbf{S}^1)$ is positively curved.

(c) For the Aloff-Wallach spaces, $\mathbf{S}^1_{k,l} \subseteq \mathrm{SU}(3)$ consists of diagonal matrices of the form $\mathrm{diag}(z^k, z^l, \bar{z}^{k+l})$, where $z \in \mathbf{S}^1 \subseteq \mathbb{C}$, and $\gcd(k, l) = 1$ ensures that there is no ineffective kernel. The inclusions $\mathbf{S}^1_{k,l} \subseteq \mathrm{U}(2) \subseteq \mathrm{SU}(3)$ lead to a homogeneous fibration

$$\mathbf{S}^3/\mathbb{Z}_{k+l} \to W^7_{k,l} \to \mathbf{CP}^2,$$

where $\mathrm{U}(2)/\mathbf{S}^1_{k,l} = \mathbf{S}^3/\mathbb{Z}_{k+l}$ is a lens space whenever $k + l \neq 0$. Assuming then $k + l \neq 0$, the computation to show that $(W^7_{k,l}, \langle\, ,\, \rangle_\lambda)$ has positive curvature (if, in addition, $kl \neq 0$) is similar to that above for the Wallach space W^6. More details may be found in [Zi2], as well as an argument showing that $W^7_{k,l}$ does not admit any homogeneous metric with positive curvature whenever $kl(k + l) = 0$.

On the other hand, the Aloff-Wallach spaces are the subfamily of the Eschenburg spaces (discussed in the next section) consisting of homogeneous spaces and, as such, positive sectional curvature follows from the arguments for the Eschenburg spaces below.

6. Biquotients with positive sectional curvature

Except for a single seven-dimensional manifold, homeomorphic but not diffeomorphic to the unit tangent bundle $T^1\mathbf{S}^4$ of the standard four-dimensional sphere $\mathbf{S}^4$, discovered independently by Dearricott [De] and Grove, Verdiani and Ziller [GVZ] and which will not be discussed here, all other known simply connected manifolds admitting a metric with positive curvature are biquotients. They fall into two infinite families of dimensions 7 and 13, the Eschenburg and Bazaikin spaces respectively, and an isolated example in dimension 6, the inhomogeneous flag manifold:

(a) Eschenburg spaces: For $p, q \in \mathbb{Z}^3$,

$$E^7_{p,q} = (\mathrm{SU}(3), g_\lambda) /\!\!/ \mathbf{S}^1_{p,q}.$$

(b) Bazaikin spaces: For $q_1, \dots, q_5 \in \mathbb{Z}$,

$$B^{13}_{q_1,\dots,q_5} = (\mathrm{SU}(5), g_\lambda) /\!\!/ (\mathrm{Sp}(2) \cdot \mathbf{S}^1_{q_1,\dots,q_5}).$$

(c) Inhomogeneous flag manifold:

$$E^6 = (\mathrm{SU}(3), g_\lambda) /\!\!/ T^2.$$

Given $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3) \in \mathbb{Z}^3$, with $\sum p_i = \sum q_i$, the Eschenburg biquotients (see [AW], [Es1]) are defined as $E^7_{p,q} := (\mathrm{SU}(3), g_\lambda) /\!\!/ \mathbf{S}^1_{p,q}$, where $\mathbf{S}^1_{p,q}$ acts isometrically on $(G, g_\lambda) = (\mathrm{SU}(3), g_\lambda)$ via

$$z \star A = \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \cdot A \cdot \mathrm{diag}(\bar{z}^{q_1}, \bar{z}^{q_2}, \bar{z}^{q_3}), \quad A \in \mathrm{SU}(3), z \in \mathbf{S}^1,$$

and the left-invariant, right $\mathrm{U}(2)$-invariant metric g_λ on $G = \mathrm{SU}(3)$ is defined by a Cheeger deformation in the direction of $K = \mathrm{U}(2) \subseteq \mathrm{SU}(3)$ of the bi-invariant metric $\langle v, w\rangle_0 = -\mathrm{Re}(\mathrm{tr}(vw))$, $v, w \in \mathfrak{g}$, where the inclusion is via

$$A \in \mathrm{U}(2) \mapsto \mathrm{diag}(A, \overline{\det(A)}) \in \mathrm{SU}(3).$$

The action is free if and only if

$$\gcd(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) = 1 \ \text{ for all permutations } \ \sigma \in S_3, \tag{6.1}$$

in which case $E^7_{p,q}$ is called an *Eschenburg space*.

It is important to remark that the above defined circle subgroup $\mathbf{S}^1_{p,q}$ is not, in general, a subgroup of $\mathrm{SU}(3) \times \mathrm{SU}(3)$. Indeed, $\mathbf{S}^1_{p,q} \subset \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3)) := \{(A, B) \in \mathrm{U}(3) \times \mathrm{U}(3) \mid \det A = \det B\}$. This is not a problem, however, since the bi-invariant metric on $\mathrm{SU}(3)$ can be thought of as the restriction of the analogously defined bi-invariant metric on $\mathrm{U}(3)$. Hence, an element of $(A, B) \in \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3))$ maps $(\mathrm{SU}(3), \langle\, , \rangle_0)$ isometrically to itself via $X \in \mathrm{SU}(3) \mapsto AXB^{-1}$. In particular, conjugation by an element of the centre of $\mathrm{U}(3)$ is an isometry (namely, the identity map) of $(\mathrm{SU}(3), \langle\, , \rangle_0)$, and remains an isometry with respect to the new metric g_λ. Therefore the Eschenburg biquotient $E^7_{p',q'}$ defined by the action of the circle $\mathbf{S}^1_{p',q'}$, where $p' = (p_1+c, p_2+c, p_3+c)$ and $q' = (q_1+c, q_2+c, q_3+c)$, with $c \in \mathbb{Z}$, is isometric to $E^7_{p,q}$. Furthermore, introducing an ineffective kernel to the circle action will not alter the isometry class of the biquotient. Thus $E^7_{\tilde{p},\tilde{q}}$ defined by $\tilde{p} = (kp_1, kp_2, kp_3)$ and $\tilde{q} = (kq_1, kq_2, kq_3)$ is isometric to $E^7_{p,q}$. In particular, it follows that a circle action by $\mathbf{S}^1_{p,q} \subset \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3))$ can then be rewritten as the action of a circle subgroup of $\mathrm{SU}(3) \times \mathrm{SU}(3)$ via the change of parameters $(p_1, p_2, p_3, q_1, q_2, q_3) \mapsto (3p_1 - \kappa, 3p_2 - \kappa, 3p_3 - \kappa, 3q_1 - \kappa, 3q_2 - \kappa, 3q_3 - \kappa)$, where $\kappa := \sum p_i = \sum q_i$, without changing the isometry class.

From Section 4 it is clear that, for the $\mathbf{S}^1_{p,q}$-action, permuting the p_i (via the action of the Weyl group of $\mathrm{SU}(3)$) and permuting q_1, q_2 are isometries, while permuting all of the p_i and swapping p, q are diffeomorphisms. Indeed, given the fixed choice of embedding $\mathrm{U}(2) \hookrightarrow \mathrm{SU}(3)$, cyclic permutations of the q_i (and, similarly, swapping p and q and considering cyclic permutations of the p_i) induce, in general, non-isometric metrics on the quotient $E^7_{p,q}$.

Lemma 6.1 ([Es1]). *Let*

$$y_1 := \mathrm{diag}(-2i, i, i), \quad y_3 := \mathrm{diag}(i, i, -2i) \in \mathfrak{g} = \mathfrak{su}(3).$$

Then a plane $\sigma \subset \mathfrak{g} = \mathfrak{su}(3)$ has zero curvature with respect to g_λ if and only if either $y_3 \in \sigma$, or $\mathrm{Ad}_k\, y_1 \in \sigma$ for some $k \in K$.

Proof. Notice first that there is a decompostion $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{su}(2)$ of the Lie algebra of $K = \mathrm{U}(2)$, where $\mathfrak{z} = \{t y_3 \mid t \in \mathbb{R}\}$ denotes the Lie algebra of the centre of $K \subseteq G$ and $\mathfrak{su}(2)$ the Lie algebra of $\mathrm{SU}(2) \subseteq \mathrm{U}(2)$.

By Lemma 3.1, if $x, y \in \mathfrak{g}$ span σ then, since $K \subseteq G$ is a symmetric pair of rank 1, the $\mathfrak{p}$-components of x and y must be linearly dependent. Hence, without loss of generality, $x \in \mathfrak{k}$. On the other hand, as the $\mathfrak{k}$-components of x and y must also commute and $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{su}(2)$, it follows that the $\mathfrak{su}(2)$-components of x and y commute, which is only possible if they are linearly dependent. Therefore, again without loss of generality, either $x \in \mathfrak{z}$ and $y \in \mathfrak{p} \oplus \mathfrak{su}(2)$, or else $x \in \mathfrak{k}$ and $y \in \mathfrak{z} \oplus \mathfrak{p}$.

Since $[x, y] = 0$ by Lemma 3.1, in the first case it follows that $x \in \mathfrak{z}$ and $y \in \mathfrak{su}(2)$, whereas in the second case one easily sees that (up to scaling) x must have two eigenvalues i, hence must be conjugate via an element of K to y_1, i.e. $x = \mathrm{Ad}_k\, y_1$ for some $k \in K$.

Conversely, for dimension reasons there exist vectors such that in either case the identities in Lemma 3.1 are satisfied. $\square$

Theorem 6.2 **([Es1]).** *There exists an ordering on the q_i such that $E^7_{p,q} := (\mathrm{SU}(3), g_\lambda)/\!\!/\mathbf{S}^1_{p,q}$ has positive curvature if and only if*

$$q_i \notin [\underline{p}, \overline{p}] \ \text{ for } i = 1, 2, 3, \tag{6.2}$$

where $\underline{p} := \min\{p_1, p_2, p_3\}$, $\overline{p} := \max\{p_1, p_2, p_3\}$.

Proof. Suppose $q_i \notin [\underline{p}, \overline{p}]$ for $i = 1, 2, 3$ and let $P = i\,\mathrm{diag}(p_1, p_2, p_3)$, $Q = i\,\mathrm{diag}(q_1, q_2, q_3)$. The vertical subspace at $A = (a_{ij}) \in SU(3)$, left-translated to $\mathrm{Id} \in \mathrm{SU}(3)$, is

$$(dL_{A^*})_A \mathcal{V}_A = \{t\, v_A \mid t \in \mathbb{R},\ v_A := \mathrm{Ad}_{A^*} P - Q\},$$

where $A^* = \bar{A}^t$. Notice that $y_3 \in \mathfrak{k}$. Thus $0 = g_\lambda(v_A, y_3)$ if and only if $0 = \langle v_A, y_3 \rangle_0$. Now, since $\langle v, w \rangle_0 = -\operatorname{Re}\operatorname{tr}(vw)$,

$$0 = g_\lambda(v_A, y_3) \iff \sum_{j=1}^{3} |a_{j3}|^2 p_j = q_3. \tag{6.3}$$

Similarly, for $\mathrm{Ad}_k\, y_1$, some $k \in K$,

$$0 = g_\lambda(v_A, \mathrm{Ad}_k\, y_1) \iff \sum_{j=1}^{3} |(Ak)_{j1}|^2 p_j = |k_{11}|^2 q_1 + |k_{21}|^2 q_2. \tag{6.4}$$

Since $q_i \notin [\underline{p}, \overline{p}]$, $i = 1, 2, 3$, and $\sum p_j = \sum q_j$, two of the q_i must lie on one side of $[\underline{p}, \overline{p}]$, and one on the other. Reorder and relabel the q_i so that q_1, q_2 lie on the same side of the interval $[\underline{p}, \overline{p}]$. Since A and k are both unitary there are no solutions to either (6.3) or (6.4). Hence $E^7_{p,q}$ has positive curvature.

For the converse suppose that $E^7_{p,q}$ has positive curvature. If $q_i \in [\underline{p}, \overline{p}]$ for some $i = 1, 2, 3$ then by continuity there exists a solution to either (6.3) or (6.4), and hence either y_3 or $\mathrm{Ad}_k\, y_1$ is horizontal. By Lemma 6.1, since the orbits of $\mathbf{S}^1_{p,q}$ are one-dimensional, one can always find another horizontal vector x which, together with either y_3 or $\mathrm{Ad}_k\, y_1$, will span a zero-curvature plane. The main result of [Ta] then implies that this horizontal zero-curvature plane must project to a zero-curvature plane in $E^7_{p,q}$, which is a contradiction. $\square$

Corollary 6.3. *The inhomogeneous flag manifold $E^6 = (\mathrm{SU}(3), g_\lambda)/\!\!/T^2$ is positively curved, where T^2 acts freely and isometrically on $(\mathrm{SU}(3), g_\lambda)$ via*

$$(z,w)\cdot A = \mathrm{diag}(z,w,zw)A\,\mathrm{diag}(1,1,\bar z^2\bar w^2),\ \ \textit{for}\ \ (z,w)\in T^2,\ A\in \mathrm{SU}(3).$$

Proof. Consider the circle subgroup of T^2 with slope (k,l), where $k,l\in\mathbb{Z}$, $\gcd(k,l)=1$ and $kl>0$. The resulting circle action on $\mathrm{SU}(3)$ is given by

$$z\cdot A = \mathrm{diag}(z^k, z^l, z^{k+l})A\,\mathrm{diag}(1,1,\bar z^{2(k+l)}).$$

The corresponding biquotient is an Eschenburg space $E^7_{p,q}$ with $p=(k,l,k+l)$ and $q=(0,0,2(k+l))$. By Theorem 6.2 $E^7_{p,q}$ is positively curved. Moreover, $E^7_{p,q}$ is a principal $\mathbf{S}^1$-bundle over E^6, where the principal $\mathbf{S}^1$ action is the free, isometric action by a generating circle of T^2 complementary to the circle with slope (k,l). Therefore there is a Riemannian submersion $E^7_{p,q}\to E^6$, hence E^6 is positively curved. $\square$

In his Habilitation [Es1], Eschenburg has shown that every other free biquotient T^2 action on $\mathrm{SU}(3)$ is conjugate to either that yielding the flag manifold W^6 or that yielding the inhomogeneous flag above. As discovered in [Es2], these two manifolds are topologically distinct. Indeed, they have the same cohomology groups, but they are distinguished by their cohomology ring structures.

The topology of the positively curved Eschenburg spaces has been widely studied. Eschenburg himself showed in [Es2] that there are infinitely many homotopy types, by exploiting the order of the torsion cohomology group $H^4(E^7_{p,q};\mathbb{Z}) = \mathbb{Z}_{s(p,q)}$. If, on the other hand, one fixes the cohomology ring, then Chinburg, Escher and Ziller have shown that there are only finitely many diffeomorphism types of positively curved Eschenburg spaces. They exploited the work of Kruggel [Kr1, Kr2, Kr3], who had previously determined the topological invariants which classify a large class of the Eschenburg spaces up to homotopy, homeomorphism and diffeomorphism. Shankar [Sh] has shown that most Eschenburg spaces are strongly inhomogeneous, i.e. they are not even homotopy equivalent to a homogeneous space. He also showed that there are positively curved Eschenburg spaces which are homotopy equivalent but not homeomorphic to Aloff-Wallach spaces. Among the Aloff-Wallach spaces (see [KS]) and, more generally, the Eschenburg spaces (see [CEZ]), there are pairs of positively curved spaces which are homeomorphic but not diffeomorphic to each other. Furthermore, there are also pairs which are diffeomorphic but not isometric.

It remains only to deal with the Bazaikin spaces (see [Ba], [DE]). Given the bi-invariant metric $\langle v,w\rangle_0 = -\mathrm{Re}(\mathrm{tr}(vw))$, $v,w\in\mathfrak{g}$, on $G=\mathrm{SU}(5)$, perform a Cheeger deformation in the direction of the subgroup $K=\mathrm{U}(4)\subseteq G=\mathrm{SU}(5)$, where K is the image of the inclusion map $\mathrm{U}(4)\hookrightarrow\mathrm{SU}(5);\ A\mapsto \mathrm{diag}(A,\overline{\det A})$. The resulting metric g_λ on G is left invariant and right K-invariant.

Consider now

$$U_{q_1,\dots,q_5} := \mathrm{Sp}(2)\cdot\mathbf{S}^1_{q_1,\dots,q_5} = (\mathrm{Sp}(2)\times\mathbf{S}^1_{q_1,\dots,q_5})/\mathbb{Z}_2,\quad \mathbb{Z}_2=\{\pm(1,I)\},$$

where $q_1,\dots,q_5\in\mathbb{Z}$ and $\mathrm{Sp}(2)$ is considered as a subgroup of $\mathrm{SU}(4)$ via the standard inclusion $\mathrm{Sp}(2)\hookrightarrow\mathrm{SU}(4)$ given by

$$A = S+Tj\in\mathrm{Sp}(2)\mapsto \hat A = \begin{pmatrix} S & T\\ -\bar T & \bar S\end{pmatrix}\in\mathrm{SU}(4),\quad S,T\in M_2(\mathbb{C}). \tag{6.5}$$

Therefore $U_{q_1,\dots,q_5} = \mathrm{Sp}(2)\cdot \mathbf{S}^1_{q_1,\dots,q_5}$ acts effectively and isometrically on (G, g_λ) via

$$[A, z] \star B = \mathrm{diag}(z^{q_1}, \dots, z^{q_5}) \cdot B \cdot \mathrm{diag}(\hat{A}^{-1}, \bar{z}^q),$$

with $q := \sum q_i$, $z \in \mathbf{S}^1$, $B \in G$, and $A \in \mathrm{Sp}(2) \subseteq \mathrm{SU}(4)$.

It is not difficult to show that the action of $\mathrm{Sp}(2)\cdot\mathbf{S}^1_{q_1,\dots,q_5}$ is free if and only all $q_1, \dots, q_5$ are odd and

$$\gcd(q_{\sigma(1)} + q_{\sigma(2)}, q_{\sigma(3)} + q_{\sigma(4)}) = 2 \text{ for all permutations } \sigma \in S_5. \tag{6.6}$$

The quotient $B^{13}_{q_1,\dots,q_5} := (G, g_\lambda)/\!\!/U_{q_1,\dots,q_5}$ is hence a manifold called a *Bazaikin space*. From the discussion in Section 4 it follows that permuting the q_i is an isometry of $B^{13}_{q_1,\dots,q_5}$. Notice further that, if $q_1 = \dots = q_5 = 1$, then the resulting Bazaikin space $B^{13}_{1,1,1,1,1}$ is none other than the Berger space B^{13}.

Let $Q = i\,\mathrm{diag}(q_1, \dots, q_5)$. The vertical subspace at $A \in G$, left-translated back to $\mathrm{Id} \in G$, with respect to the $U_{q_1,\dots,q_5}$ action may be written as

$$(dL_{A^*})_A \mathcal{V}_A = \{t \,\mathrm{Ad}_{A^*} Q - \mathrm{diag}(V, itq) \mid t \in \mathbb{R}, V \in \mathfrak{sp}(2) \subseteq \mathfrak{su}(4)\},$$

where $A^* = \bar{A}^t$. One must determine when a zero-curvature plane with respect to g_λ is horizontal at $A \in G$. A vector $x = x_\mathfrak{p} + \frac{1}{\lambda} x_\mathfrak{k}$ is orthogonal to $(dL_{A^*})_A \mathcal{V}_A$ with respect to g_λ if and only if

$$\langle x, \mathrm{Ad}_{A^*} Q - \mathrm{diag}(0, 0, 0, 0, iq)\rangle_0 = 0 \quad \text{and} \quad \langle x, \mathrm{diag}(V, 0)\rangle_0 = 0, \tag{6.7}$$

for all $V \in \mathfrak{sp}(2) \subseteq \mathfrak{su}(4)$.

Lemma 6.4. *A $\sigma \subseteq \mathfrak{g}$ is a horizontal zero-curvature plane with respect to g_λ if and only if either*

$$w_1 := \mathrm{diag}(i, i, i, i, -4i) \text{ or } w_2 := \mathrm{Ad}_k \,\mathrm{diag}(2i, -3i, 2i, -3i, 2i),$$

for some $k \in \mathrm{Sp}(2) \subseteq K$, is in σ and is horizontal.

Proof. Suppose that σ has zero-curvature with respect to g_λ and is spanned by $x, y \in \mathfrak{g}$. As $K \in G$ is a compact rank 1 symmetric pair, $[x_\mathfrak{p}, y_\mathfrak{p}] = 0$ by Lemma 3.1, and hence it may be assumed without loss of generality that $y_\mathfrak{p} = 0$, i.e $x = x_\mathfrak{p} + x_\mathfrak{k}$, $y = y_\mathfrak{k}$.

If $x_\mathfrak{p} = 0$ too, then $x, y \in \mathfrak{k}$. Notice that $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{sp}(2) \oplus \mathfrak{m}$, where the decomposition is orthogonal with respect to $\langle\,,\,\rangle_0$, $\mathfrak{z}$ is the centre of $\mathfrak{k}$ and is generated by $\mathrm{diag}(i, i, i, i, -4i)$, and $\mathfrak{m} = \mathfrak{sp}(2)^\perp \subset \mathfrak{su}(4)$. By ssumption, x and y are horizontal, hence orthogonal to $\mathfrak{sp}(2)$ with respect to $\langle\,,\,\rangle_0$. Therefore $x, y \in \mathfrak{z} \oplus \mathfrak{m}$, and $[x, y] = 0$ if and only if $[x_\mathfrak{m}, y_\mathfrak{m}] = 0$. However, since $\mathrm{Sp}(2) = \mathrm{Spin}(5) \subseteq \mathrm{SU}(4) = \mathrm{Spin}(6)$ is a rank one symmetric pair, $x_\mathfrak{m}$ and $y_\mathfrak{m}$ must be linearly dependent. Thus, without loss of generality it may be assumed that $x = x_\mathfrak{m}$, $y = y_\mathfrak{z}$. In particular, $\mathfrak{z} \subseteq \sigma$, i.e. $w_1 = \mathrm{diag}(i, i, i, i, -4i) \in \sigma$.

Note that w_1 being horizontal is not only a necessary condition for $\sigma \subset \mathfrak{k}$ to be a horizontal zero-curvature plane, but is also sufficient for the existence of such a plane as, by counting dimensions, one may always find a vector $x \in \mathfrak{m}$ such that $\sigma = \{x, w_1\}$ is a horizontal zero-curvature plane.

On the other hand, suppose now that $x_\mathfrak{p} \neq 0$. The conditions for zero-curvature in Lemma 3.1 become $0 = [x_\mathfrak{p}, y_\mathfrak{k}] = [x_\mathfrak{k}, y_\mathfrak{k}]$. Suppose that

$$x_\mathfrak{p} = \begin{pmatrix} 0 & \xi \\ -\bar{\xi}^t & 0 \end{pmatrix}, \quad y = y_\mathfrak{k} = \mathrm{diag}(Z, -\mathrm{tr}(Z)),$$

where $\xi \in \mathbb{C}^4$ and $Z \in \mathfrak{u}(4) = \mathfrak{z} \oplus \mathfrak{su}(4)$. Then $0 = [x_\mathfrak{p}, y_\mathfrak{k}]$ if and only if $Z\xi = -\mathrm{tr}(Z)\xi$. Let $Z = it\,\mathrm{Id} + Z' \in \mathfrak{z} \oplus \mathfrak{su}(4)$, $t \in \mathbb{R}$. The requirement that y be horizontal implies that

$Z' \perp \mathfrak{sp}(2) \subset \mathfrak{su}(4)$. Recall once again that $\mathrm{SU}(4) = \mathrm{Spin}(6)$ and $\mathrm{Sp}(2) = \mathrm{Spin}(5)$, hence $\mathrm{SU}(4)/\mathrm{Sp}(2) = \mathbf{S}^5$ and, since $\mathrm{Sp}(2) = \mathrm{Spin}(5)$ acts transitively on distance spheres in $\mathfrak{m} = \mathfrak{sp}(2)^{\perp} \subset \mathfrak{su}(4)$, one may write

$$Z' = k \,\mathrm{diag}(is, -is, is, -is)k^{-1}, \quad \text{for some } \ k \in \mathrm{Sp}(2).$$

This in turn implies that Z may be written as

$$Z = k \,\mathrm{diag}(i(t+s), i(t-s), i(t+s), i(t-s))k^{-1}, \quad k \in \mathrm{Sp}(2).$$

However, it was established above that $\mathrm{tr}\, Z = -4it$ is an eigenvalue of Z. Therefore either $-4t = t + s$ or $-4t = t - s$, i.e. $s = -5t$ or $s = 5t$, and y must be conjugate by an element of $\mathrm{Sp}(2)$ to either $\mathrm{diag}(-4it, 6it, -4it, 6it, -4it)$ or $\mathrm{diag}(6it, -4it, 6it, -4it, -4it)$. It follows that, up to scaling,

$$y = k \,\mathrm{diag}(2i, -3i, 2i, -3i, 2i)k^{-1} \in \sigma, \quad k \in \mathrm{Sp}(2) \subseteq K \subseteq G.$$

Conversely, if such a vector y is horizontal it is not difficult to find a complementary vector x such that they span a plane σ which is horizontal and has zero curvature with respect to g_λ. Indeed, setting $x_{\mathfrak{k}} = 0$ means x is automatically orthogonal to $\mathfrak{sp}(2)$, and it remains to choose $x = x_{\mathfrak{p}}$ such that x satisfies the first condition of (6.7), namely that x is orthogonal to a one-dimensional subspace. A choice of appropriate $x = x_{\mathfrak{p}}$ is equivalent to choosing an eigenvector for Z above. The set of such eigenvectors has dimension bigger than one. Hence $x = x_{\mathfrak{p}}$ may be chosen such that it has the desired properties. □

Lemma 6.5. *The vectors*

$$w_1 = \mathrm{diag}(i, i, i, i, -4i) \quad \text{and} \quad w_2 = \mathrm{Ad}_k \,\mathrm{diag}(2i, -3i, 2i, -3i, 2i),$$

for $k \in Sp(2)$, are horizontal with respect to g_λ at $A = (a_{ij}) \in G$ if and only if

$$q = \sum_{\ell=1}^{5} |a_{\ell 5}|^2 q_\ell, \quad \text{and} \tag{6.8}$$

$$0 = \sum_{\ell=1}^{5} (|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2) q_\ell \tag{6.9}$$

respectively.

Proof. First recall that both w_1 and w_2 lie in $\mathfrak{k} = \mathfrak{u}(4)$. Therefore w_1 and w_2 are horizontal with respect to g_λ if and only if they are horizontal with respect to $\langle\, , \rangle_0$. Moreover, by the discussion in the proof of Lemma 6.4, w_1 and w_2 are both orthogonal to $\mathfrak{sp}(2)$ with respect to $\langle\, , \rangle_0$. Hence one need only obtain expressions for w_1 and w_2 being orthogonal with respect to $\langle\, , \rangle_0$ to $v_A := \mathrm{Ad}_{A^*} Q - \mathrm{diag}(0, 0, 0, 0, iq)$, where $Q = \mathrm{diag}(iq_1, \dots, iq_5)$.

Recall that $\langle x, y\rangle_0 = -\mathrm{Re}\,\mathrm{tr}(xy)$. Then w_1 is horizontal if and only if

$$\begin{aligned} -4q &= \langle \mathrm{diag}(0, 0, 0, 0, iq), w_1 \rangle_0 \\ &= \langle \mathrm{Ad}_{A^*} Q, w_1\rangle_0 \\ &= \sum_{\ell=1}^{5} (|a_{\ell 1}|^2 + |a_{\ell 2}|^2 + |a_{\ell 3}|^2 + |a_{\ell 4}|^2 - 4|a_{\ell 5}|^2) q_\ell. \end{aligned}$$

Now using the fact that A is unitary together with $q = \sum_{\ell=1}^{5} q_\ell$ yields

$$-4q = q - 5\sum_{\ell=1}^{5} |a_{\ell 5}|^2 q_\ell,$$

from which (6.8) follows immediately.

Consider now $w_2 = \operatorname{Ad}_k \widehat{w}$, where $\widehat{w} = \operatorname{diag}(2i, -3i, 2i, -3i, 2i)$. Then w_2 is horizontal if and only if

$$\begin{aligned}
2q &= \langle \operatorname{diag}(0,0,0,0,iq), \widehat{w}\rangle_0 \\
&= \langle \operatorname{Ad}_{k^*} \operatorname{diag}(0,0,0,0,iq), \widehat{w}\rangle_0 \quad \text{for } k \in \operatorname{Sp}(2) \subseteq \operatorname{SU}(4) \\
&= \langle \operatorname{diag}(0,0,0,0,iq), w_2\rangle_0 \\
&= \langle \operatorname{Ad}_{A^*} Q, w_2\rangle_0 \\
&= \left\langle \operatorname{Ad}_{(Ak)^*} Q, \widehat{w}\right\rangle_0 \\
&= \sum_{\ell=1}^{5} \left(2|(Ak)_{\ell 1}|^2 - 3|(Ak)_{\ell 2}|^2 + 2|(Ak)_{\ell 3}|^2 - 3|(Ak)_{\ell 4}|^2 + 2|(Ak)_{\ell 5}|^2\right) q_\ell \\
&= \sum_{\ell=1}^{5} \left(2 - 5\left(|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2\right)\right) q_\ell, \quad \text{since } A \text{ is unitary.}
\end{aligned}$$

Equation 6.9 now follows immediately from $q = \sum_{\ell=1}^{5} q_\ell$. □

Theorem 6.6. *The Bazaikin space* $B^{13}_{q_1,\dots,q_5} = (\operatorname{SU}(5), g_\lambda) /\!\!/ \operatorname{Sp}(2) \cdot \mathbf{S}^1_{q_1,\dots,q_5}$ *is positively curved if and only if*

$$q_{\sigma(1)} + q_{\sigma(2)} > 0 \ \ (\text{or } < 0) \quad \textit{for all permutations } \sigma \in S_5. \tag{6.10}$$

Proof. Suppose $q_{\sigma(1)} + q_{\sigma(2)} > 0$ for all permutations $\sigma \in S_5$. In particular notice that this implies that $q > 0$. By Lemmas 6.4 and 6.5, one need only examine equations (6.8) and (6.9) to obtain the desired result. Consider first equation (6.8) in the alternative form

$$\sum_{\ell=1}^{5} (1 - |a_{\ell 5}|^2) q_\ell = 0.$$

Since $A \in \operatorname{SU}(5)$ is unitary, either $1 - |a_{\ell 5}|^2 \neq 0$ for all $\ell = 1, \dots, 5$, or $1 - |a_{\ell_0 5}|^2 = 0$ for exactly one $\ell_0 \in \{1, \dots, 5\}$. In the first case, for some $\ell_0 \in \{1, \dots, 5\}$,

$$\sum_{\ell=1}^{5} (1 - |a_{\ell 5}|^2) q_\ell \geqslant (1 - |a_{\ell_0 5}|^2) \sum_{\ell=1}^{5} q_\ell = (1 - |a_{\ell_0 5}|^2) q > 0,$$

and so there are no solutions to equation (6.8). In the second case equation (6.8) reduces, without loss of generality, to $q_1 = q$. Thus, in order to have solutions, one requires $q_2 + q_3 + q_4 + q_5 = 0$, which is impossible by hypothesis. Hence there can be no solutions to equation (6.8).

Consider now equation (6.9). Since A is unitary there must be at least two indices $\ell \in \{1, \dots, 5\}$ such that $|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2 \neq 0$. Without loss of generality, assume that $\ell = 1$ gives the minimal $|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2 \neq 0$. Then, defining $\hat{q}$ to be the sum of those q_j for which $|(Ak)_{j2}|^2 + |(Ak)_{j4}|^2 \neq 0$,

$$\sum_{\ell=1}^{5} \left(|(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 4}|^2\right) q_\ell \geqslant \left(|(Ak)_{12}|^2 + |(Ak)_{14}|^2\right) \hat{q} > 0$$

since $\hat{q}$ is the sum of at least two q_ℓ and must therefore be positive by hypothesis. Hence equation (6.9) has no solutions.

Conversely, say now that $B^{13}_{q_1,\dots,q_5}$ admits positive curvature. Suppose, without loss of generality, that $q_1 + q_2 \leqslant 0$ and $q_2 + q_3 > 0$. If $q_1 + q_2 = 0$ then choosing $A \in \operatorname{SU}(5)$ such that $|a_{15}|^2 = |a_{25}|^2 = \frac{1}{2}$ yields a solution of equation (6.8) and, by Lemmas 6.4 and 6.5, there exists a horizontal zero-curvature plane at this A, hence by [Ta] at the image point in $B^{13}_{q_1,\ldots,q_5}$, which is a contradiction. On the other hand, if $q_1 + q_2 < 0$ and $q_2 + q_3 > 0$ then, since both $A \in \mathrm{SU}(5)$ and $k \in \mathrm{Sp}(2) \subseteq K$ are unitary, a pair of matrices A_0 and k_0 may be chosen such that $|(A_0k_0)_{12}|^2 = 1$ and $|(A_0k_0)_{24}|^2 = 1$, and similarly A_1 and k_1 such that $|(A_1k_1)_{22}|^2 = 1$ and $|(A_1k_1)_{34}|^2 = 1$. Therefore, by connectedness,

$$q_1 + q_2 \leqslant \sum_{\ell=1}^{5}(|(A_tk_t)_{\ell 2}|^2 + |(A_tk_t)_{\ell 4}|^2)q_\ell \leqslant q_2 + q_3$$

along some path A_tk_t in $\mathrm{SU}(5)$, $t \in [0,1]$, from A_0k_0 to A_1k_1. By continuity, one obtains a solution to equation (6.9) and hence a zero-curvature plane by Lemmas 6.4 and 6.5. □

As for the Eschenburg spaces, Bazaikin spaces can be distinguished by the torsion in their cohomology. Indeed, the fact that $H^6(B^{13}_{q_1,\ldots,q_5};\mathbb{Z}) = H^8(B^{13}_{q_1,\ldots,q_5};\mathbb{Z}) = \mathbb{Z}_{s(q_1,\ldots,q_5)}$ allows one to conclude that there are infinitely many homotopy types of Bazaikin spaces admitting positive curvature ([Ba], [FZ]). Moreover, Florit and Ziller [FZ] have shown that if the cohomology ring is fixed, there are only finitely many diffeomorphism types with positive curvature. However, in contrast to the case of the Eschenburg spaces, they have shown that there is strong evidence to suggest that positively curved Bazaikin spaces are pairwise homeomorphically distinct. They have also shown that among the positively curved Bazaikin spaces, only the Berger space B^{13} can be homotopically equivalent to a homogeneous space. It should be noted, however, that there is no classification of Bazaikin spaces corresponding to that achieved by Kruggel for the Eschenburg spaces.

Finally, Taimanov [Tai] showed, that every Bazaikin space contains an Eschenburg space as a totally geodesic submanifold. Eschenburg and Dearricott [DE] subsequently showed that there are, in fact, generically ten totally geodesically embedded Eschenburg spaces in each Bazaikin space. The Bazaikin space has positive curvature if and only if each of these totally geodesic Eschenburg spaces does.

Conversely, it was shown in [Ke] that, given any Eschenburg space, there are infinitely many topologically distinct Bazaikin spaces into which it can be totally geodesically embedded. It remains, however, unknown whether every positively curved Eschenburg space can be totally geodesically embedded into a positively curved Bazaikin space. In fact, for the usual construction of these totally geodesic embeddings, there exist infinitely many counter-examples.

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