Semester I Examinations 2017-18

Exam Code(s)	1BCT1, 1BMS1, 1BPC1, 1BPT1, 1BS1 1EH1, 1FM1, 1HF1, 1MR1, 1BME1
Exam	First Year
Module	MATHEMATICS
Module Code	MA180-1 & MA185-1 & MA190-I
External Examiner	Prof T. Brady
Internal Examiner	Prof G. Ellis*
Instructions	Answer all six questions.
Duration	2 hours
No. of Pages	3 pages (including this cover page)
Discipline	Mathematics

Requirements:

Release to Library:	Yes
Other Materials	Non-programmable calculators

1.

(a) The enciphered message

$$HVVH$$

was produced by applying the enciphering function

$$f_E\colon \mathbb{Z}_{37} \to \mathbb{Z}_{37}, x \mapsto 14x + 20$$

to single letter message units over the 37-letter alphabet

$$0, ..., 9, A = 10, B = 11, \ldots, Z = 35, \_ = 36\ .$$

(i) [3 marks] Find the inverse of 14 modulo 37.
(ii) [2 marks] Determine the corresponding deciphering function.
(iii) [3 marks] Decipher the message.

(b) (i) [3 marks] Factorise 360 as a product of primes.
(ii) [3 marks] Then calculate $\phi(360)$, which is the number of integers from 1 to 360 that are coprime to 360.
(iii) [3 marks] Finally, calculate

$$11^{98} \mod 360.$$

2.

(a) [8 marks] The ciphertext

$$ESDCWNMH$$

was produced by applying the function

$$f_E\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{with } A = \begin{pmatrix} 2 & 3 \\ 7 & 5 \end{pmatrix}$$

to 2-letter message units over the alphabet $A = 0, B = 1, ..., Z = 25$. Use

$$A^{-1} = \begin{pmatrix} 9 & 5 \\ 3 & 14 \end{pmatrix} \mod 26$$

to determine the first **four** letters of plaintext.

(b) [8 marks] Use row operations to find the inverse of

$$B = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix}.$$

3.

(a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = -x$ and let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be anticlockwise rotation through $90°$ about the origin.

(i) [4 marks] Find the matrices of f and g with respect to the standard basis vectors.
(ii) [4 marks] Find the matrix of the composite transformation $g \circ f\colon \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the standard basis vectors.

(b) [9 marks] Consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

Verify that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ are eigenvectors of A. Hence find a diagonal matrix D and matrix E such that $A^n = ED^nE^{-1}$.

4.

(a) Calculate the following limits:

$$\text{(i) [4 marks]} \quad \lim_{x\to 0} \frac{\sqrt{(4+x)}-2}{x} \qquad \text{and} \qquad \text{(ii) [4 marks]} \quad \lim_{x\to+\infty} \frac{1}{\sqrt{(x^2+2x)}-x}\,.$$

(b) [9 marks] For what values of k and ℓ is the following function $g\colon \mathbb{R} \to \mathbb{R}$ continuous and differentiable at all points?

$$g(x) = \begin{cases} x^3 + kx^2 + x\,, & \text{if } x < 1 \\ \ell x + k\,, & \text{if } x \geq 1 \end{cases}$$

5.

(a) [7 marks] State the Intermediate Value Theorem and use it to prove that the equation

$$x^3 - 4x - 1 = 0$$

has three real solutions.

(b) Consider the function

$$f(x) = x^4 - 4x^3 + 10.$$

(i) [3 marks] Find all critical points of f.
(ii) [2 marks] Find the intervals on which f increases/decreases.
(iii) [3 marks] For each critical point, decide whether it is a local maximum, a local minimum, or neither.
(v) [2 marks] Find the interval on which $f(x)$ is concave down.

6.

(a) [9 marks] Find antiderivatives of each of the following functions

$$f(t) = \sqrt{t}\,, \qquad g(t) = t\,\sin(t^2)\,, \qquad h(t) = \frac{1}{2t+1}\,.$$

(b) [7 marks] The number $y(t)$ of ants at time t in a new ant colony is described by the differential equation

$$\frac{dy}{dt} = k\,y,$$

where t is measured in weeks and where k is some constant. Suppose that the colony has 1000 ants at time $t = 0$, and 2000 ants at time $t = 1$. Calculate the number of ants after 4 weeks.