What plants know about mathematics, or how evolution discovers theorems

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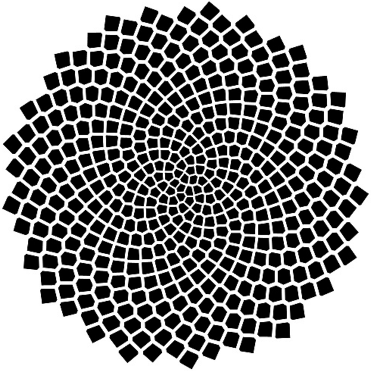
Sunflowers

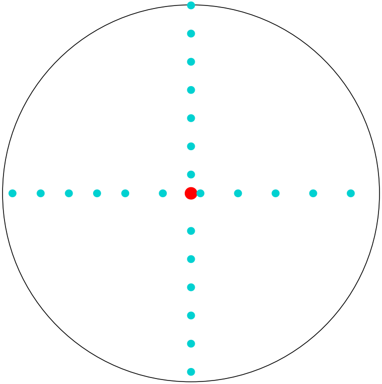


Phyllotaxis - the arrangement of seeds in the seed head

The mechanics work like this:

- A seed is laid down at the centre;
- After a rotation through an angle of θ, another one is laid down;
- This process repeats;
- Meanwhile, all seeds move radially outwards as the seed head grows from the centre.

Example $\theta = 90^\circ = \frac{1}{4} \times 360^\circ$



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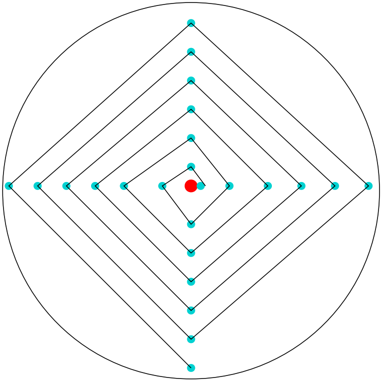

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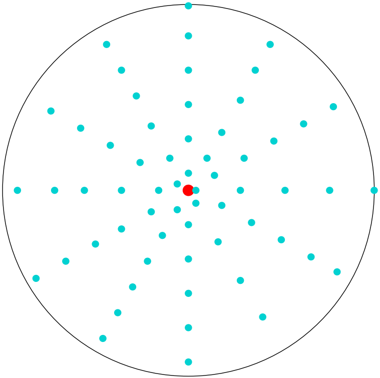
Example $\theta = 150^\circ = \frac{5}{12} \times 360^\circ$



Phyllotaxis - the arrangement of seeds in the seed head

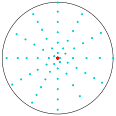
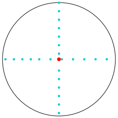
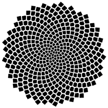
Desirable features (for the plant)

- The seeds should fill the seed head as much as possible, to avoid wasted space.
- The seeds should be densely packed, for structure and shelter.
- Seeds should be exposed to varied conditions of light, moisture etc.

The mathematical question is *what value of θ (as a fraction of the full 360°) is best for packing seeds into the seed head?*

What is the best value of θ?

1. If $\theta = \frac{5}{7} \times 360^\circ$, the seeds will all be along seven spokes radiating from the centre - not great.
2. If $\theta = \frac{1}{31} \times 360^\circ$, there will be 31 “spokes” of seeds - better than seven maybe but has the same basic problems.
3. To avoid the siutation where the seeds only occur along “spokes” the number θ needs to be irrational. This means that no more than one seed will lie on a single spoke.
4. A number is irrational if it is not a fraction involving two whole numbers. Examples include π, e, $\sqrt{2}$, $\sqrt{5}$.

The best value of θ is the one that nature found

In the sunflower seed head, and in many other botanical examples, the angle θ is

$$\frac{2}{1+\sqrt{5}} \times 360^\circ.$$

This is approximately 222.5° (or 137.5° if you measure it the other way).

The number $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is a very special and famous number, often referred to as the golden ratio or golden mean. It is famous for its many and diverse appearances in biology, but what makes it mathematically special was discovered only in 1891. It is, in a very precise sense, the most irrational number.

Rational Approximation - Hurwitz's Theorem (1891)

For calculations, we approximate irrational numbers by rational numbers that are close to them. For example, $\frac{22}{7}$ is a very good approximation to π - better than 3.14 which is $\frac{107}{50}$.

You can get better rational approximations to π if you are prepared to let the denominators in your fractions get bigger.

Examples π is quite well approximated by rational numbers with small denominators, but ϕ is not (by comparison).

Number	Rational Approximation	Error
π	$\frac{22}{7}$	≈ 0.00126
ϕ	$\frac{13}{8}$	≈ 0.00697
π	$\frac{355}{113}$	≈ 0.000000226
ϕ	$\frac{233}{144}$	≈ 0.0000216

Rational Approximation - Hurwitz's Theorem (1891)

Hurwitz was able to quantify how well (or poorly) an irrational number can be approximated by fractions with denominator not exceeding a specified value. More importantly, he proved that in this context, the approximation errors are worst for the number ϕ.

Adolf Hurwitz
(1859-1919)

In the sunflower, if the angle θ was chosen to be $\frac{1}{\pi} \times 360^{\circ}$, the pattern would initially resemble something with 22 spokes - because $\frac{7}{22}$ is a good rational approximation to $\frac{1}{\pi}$.

Proof by evolution

Hurwitz's Theorem: *The best rational approximations $\frac{p}{q}$ to an irrational number have error at most $\frac{1}{\sqrt{5}q^2}$ - but maybe much less. In the case of $\frac{1+\sqrt{5}}{2}$, this error cannot be improved upon.*

Choosing $\theta = \frac{1}{\phi} \times 360^\circ$ (as sunflowers do) is THE BEST for uniform packing of the seed head. The rational approximations to ϕ are so poor that the resulting pattern is as far away as possible from any collection of "spokes".

This difficult theorem of number theory was discovered by evolution long before it was established by mathematicians.