No lecture on Wednesday 17 March!

<u>Linear</u> <u>Algebra</u>

There is no linear isomorphism $\phi : \mathbb{R} \to \mathbb{R}^2$.

<u>Topology</u>

There is no homeomorphism $\psi : \mathbb{R} \to \mathbb{R}^2$.
(To see this, note $\mathbb{R} \setminus \{0\}$ is not connected, whereas $\mathbb{R}^2 \setminus \{\psi(0)\}$ is connected.)

<u>Linear</u> <u>Algebra</u>

There is no linear surjection $\phi : \mathbb{R} \to \mathbb{R}^2$.
(To see this

$$\dim(\operatorname{im} \phi) + \dim(\ker \phi) = \underset{1}{\underset{\shortparallel}{\dim(\mathbb{R})}}$$

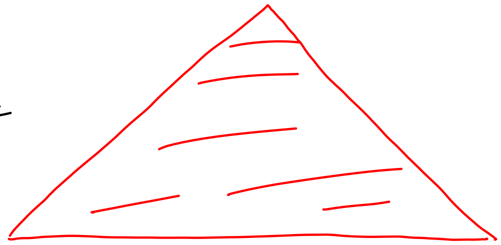
So $\dim(\operatorname{im} \phi) \leq 1$.
But $\dim(\mathbb{R}^2) = 2$.)

<u>Topology</u>

There is a surjective continuous function $\psi : \mathbb{R} \to \mathbb{R}^2$. Such a ψ is called a <u>space filling</u> curve.

Aim: we'll construct an example of a space filling curve.

Let $\triangle$ be an equilateral triangular region of $\mathbb{R}^2 = \mathbb{E}^2$ of side 1.

$\Delta =$  $\subseteq \mathbb{R}^2$

<u>Theorem</u> (Peano) There exists a surjective continuous function

$$f : [0,1] \longrightarrow \Delta .$$

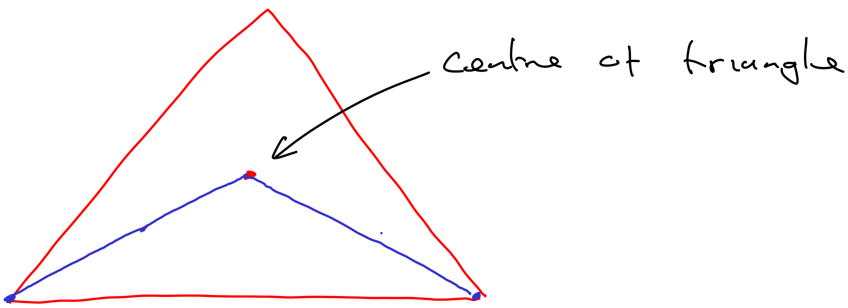
<u>Proof</u> We first construct a sequence of continuous functions

$$f_1 : [0,1] \longrightarrow \Delta$$
$$f_2 : [0,1] \longrightarrow \Delta$$
$$f_3 : [0,1] \longrightarrow \Delta$$
$$\vdots$$

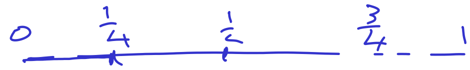
We'll take f to be the "limit" of the sequence of functions, and we'll convince ourselves that this limit is continuous and surjective.

$f_1(t)$

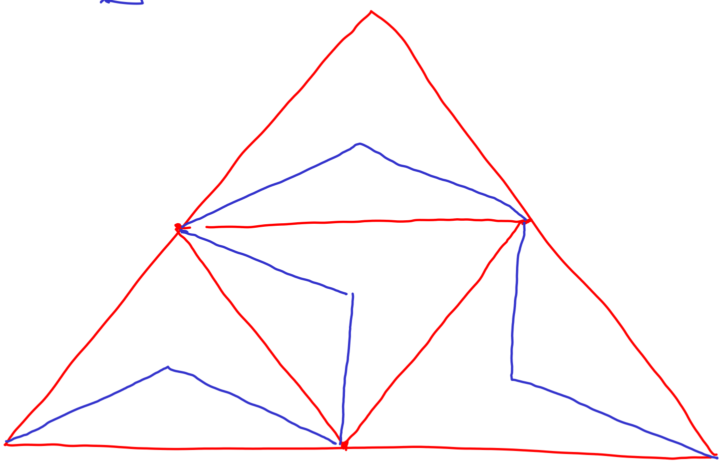
$f_1 : [0,1] \to \Delta$



$f_2(t)$

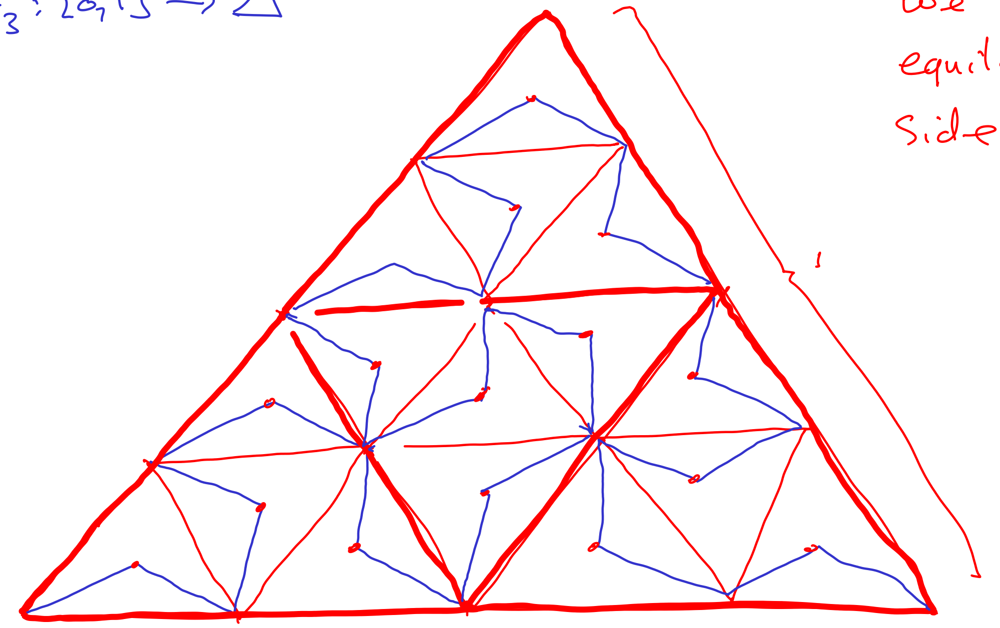


$f_2 : [0,1] \to \Delta$



$f_3(t)$

$f_3 : [0,1] \to \Delta$



we have 16 small equilateral triangles of side $\frac{1}{4}$.

for $f_n : [0,1] \longrightarrow \Delta$ we subdivide Δ into 4^{n-1} small red triangles, and the image of f_n inside each small triangle looks like f_1

To complete the proof of Peano's theorem:

1) Define f to be the limit of $f_1, f_2, f_3, \ldots$

2) Need to prove that the limit function is continuous and surjective (compactness used here).

Aside: Some basics on limits

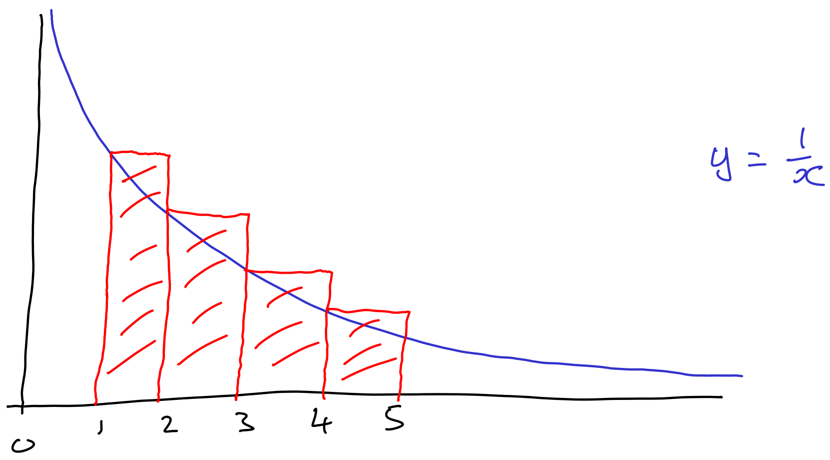
Consider

$a_1 = 1,\ a_2 = 1\frac{1}{2},\ a_3 = 1\frac{5}{6},\ \ldots,\ a_n = a_{n-1} + \frac{1}{n}, \ldots$

So $\lim\limits_{n\to\infty} a_n - a_{n-1} = \lim\limits_{n\to\infty} \frac{1}{n} = 0.$

$\lim\limits_{n\to\infty} a_n =$ does not exist.

To see why:



$a_n =$ area of first n boxes

$$> \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

Thus $\lim\limits_{n\to\infty} a_n$ does not exist.

Defn A sequence of points $a_1, a_2, a_3, \ldots$ in $\mathbb{E}^k$ is said to be a Cauchy sequence if, for any $\varepsilon > 0$ there exists some integer N such that

$$\|a_m - a_n\| < \varepsilon$$

for all $m, n > N$.

Theorem Any Cauchy sequence $a_1, a_2, \ldots$ in $\mathbb{E}^k$ has a limit $\lim_{n\to\infty} a_n$.