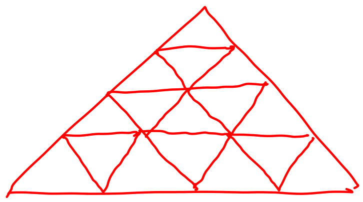
Last time we were constructing

$$f : [0,1] \longrightarrow \triangle$$

as

$$f(t) = \lim_{n\to\infty} f_n(t) .$$

Here $f_n(t)$ was defined subdividing $\triangle$ into small subtriangles



of side $\frac{1}{2^{n-1}}$. For fixed $t \in [0,1]$ the sequence

$f_1(t), f_2(t), f_3(t), \ldots$ is a Cauchy sequence, and

hence converges by the theorem given at the end of last lecture.

If t is close to t' then $f(t)$ is close to $f(t')$ and thus, intuitively, the function f is continuous. It's not difficult to convert this to a rigorous proof of continuity.

It remains to prove that f is surjective.

To prove this we'll use compactness of $[0,1]$ (see second Okuson homework) and some "related results".

Some Theory

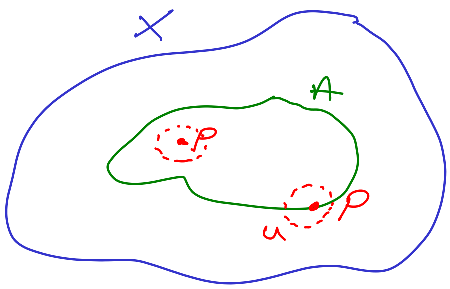
Defn Let X be a topological space. A subset $A \subseteq X$ is said to be closed if its complement $X \setminus A$ is open.

Example The subset $A = [0,1]$ of $\mathbb{R}$ (usual Euclidean topology) is a closed subset. This is because

$$\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$$

Example $(0,1]$ is neither open nor closed.

Defn Let A be a subset of a topological space X. A $p \in X$ is an accumulation point of A if every open subset U of X containing p also contains some point in $A \setminus \{p\}$.



Example Let $X = \mathbb{R}$, usual Euclidean topology, and $A = (0,1]$. Then every point in A is an accumulation point of A. Also 0 is an accumulation point even though $0 \notin A$.

Example Let $X = \mathbb{R}$, usual topology.

Consider

$$A = \left\{ \frac{1}{n} \right\}_{n=1,2,3,4,\ldots}$$

In this example 0 is the only accumulation point.

Proposition A subset A of a topological space X is closed if, and only if, it contains all its accumulation points.

Proof suppose A is closed. Then $X \setminus A$ is open. Any point $x \in X\setminus A$ lies in the open set $X \setminus A$. So no $x \in X \setminus A$ can be an accumulation point, so any accumulation point must lie in A.

Conversely, suppose A contains all of its accumulation points. Let $x \in X \setminus A$. Since x is not an accumulation point, we can find an open set

$x \in U_x \subseteq X \setminus A$.

So

$$X \setminus A = \bigcup_{x \in X\setminus A} U_x .$$

This is a union of open sets. Hence $X \setminus A$ is open. Hence A is closed. ∎