<u>Aim</u>: Show that Peano's function

$$f : [0,1] \longrightarrow \Delta$$

is surjective.

It should be clear that Δ equals the image of f together with the accumulation points of Image(f).

So we just need to show that image(f) contains all its accumulation points.

i.e. we just need to show that image(f) is closed.

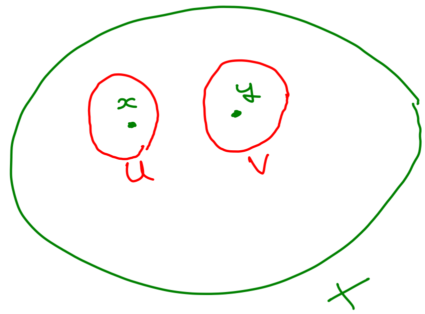
We know that $[0,1]$ is compact.
Hence (by one of the propositions proved in a previous lecture) we know that $\underbrace{f([0,1])}_{\text{Image}(f)}$ is compact.

$$f : [0,1] \twoheadrightarrow \text{Image}(f) \subseteq \Delta$$

So we just need to show that "compact" implies "closed" under a suitable hypothesis.

<u>Defn</u> A topological space X is said to be <u>Hausdorff</u> if for any distinct $x, y \in X$ there exists open sets U, V in X such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset .$$



<u>Example</u> $\mathbb{R}$ with usual Euclidean topology is Hausdorff.

<u>Example</u> $\mathbb{R}^2$ with usual topology is Hausdorff.

<u>Example</u> $\mathbb{Z}$ with cofinite topology (i.e. $U \subseteq \mathbb{Z}$ is open iff $\mathbb{Z} \setminus U$ is finite or $U = \emptyset$) is <u>not</u> Hausdorff.

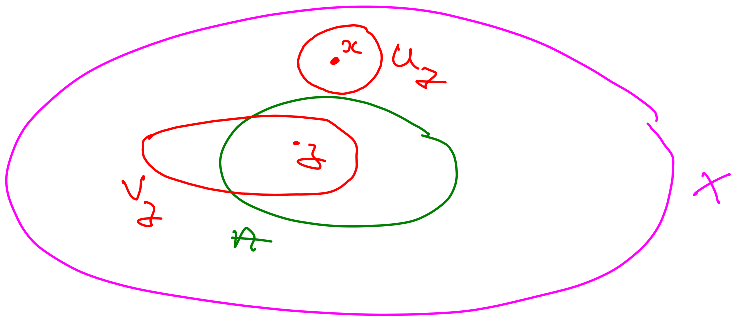
Exercise If X is Hausdorff and if Y is homeomorphic to X then Y is Hausdorff, i.e. Hausdorff is a topological property.

Theorem A compact subset of a Hausdorff topological space is closed.

Proof Let X be a Hausdorff space.
Let A be a compact subset of X.
We just need to prove that A contains all its accumulation points.

Let $x \in X \setminus A$. Just need to show that this (arbitrary) x is not an accumulation point.

Let $z \in A$. Since X is Hausdorff we can find open sets U_z, V_z in X with $z \in U_z$, $x \in V_z$ and $U \cap V = \emptyset$.



We have a collection of open sets

$$\{V_z\}_{z \in A}.$$

But A is compact, so (!) we can find a finite collection of points

$z_1, z_2, \dots, z_k \in A$

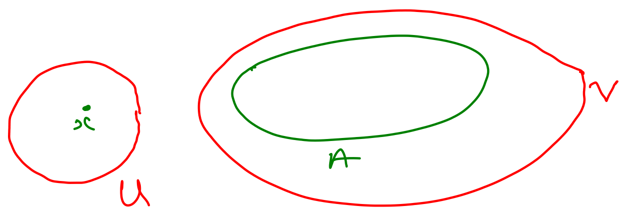
with

$A \subseteq V_{z_1} \cup V_{z_2} \cup \dots \cup V_{z_k} =: V$

Now V is disjoint from the following finite intersection

$U := U_{z_1} \cap U_{z_2} \cap \dots \cap U_{z_k}$

But U is a finite intersection of open sets and is thus open.



Since U is open we see that x is <u>not</u> an accumulation point of A. Hence A contains all its accumulation points and is therefore closed. $\blacksquare$