So far:

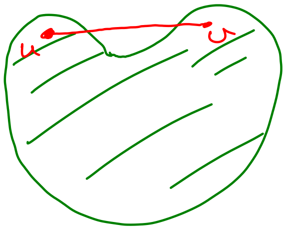
- topology has some modern application
- topology involves subtle mathematics

Next aims:

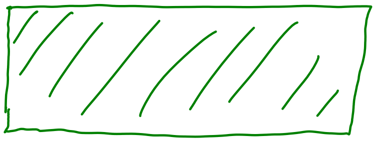
- precise definition of the Euler characteristic $\chi(X)$ of a topological space
- flavour of the ingredients of the proof that $\chi(X)$ is a topological invariant
- Application of the Euler characteristic to Economics.

Defn A set $X \subseteq \mathbb{R}^n$ is said to be **convex** if, for $u, v \in X$, the straight line from u to v lies entirely in the set.

Example $(n=2)$

$X =$ 

$\subseteq \mathbb{R}^2$ Not Convex

$X =$  $\subseteq \mathbb{R}^2$ IS Convex

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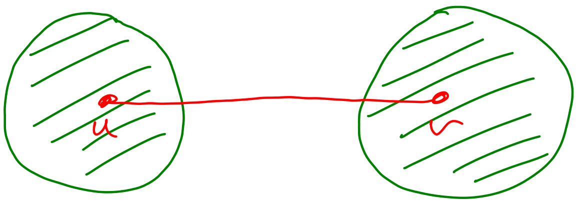
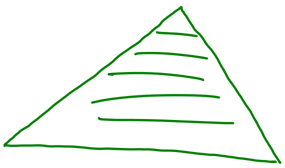
$X =$ 

$\subseteq \mathbb{R}^2$ Not Convex

$X =$  $\subseteq \mathbb{R}^2$ IS Convex

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Suppose $v_0, v_1, \dots, v_k \in \mathbb{R}^n$.

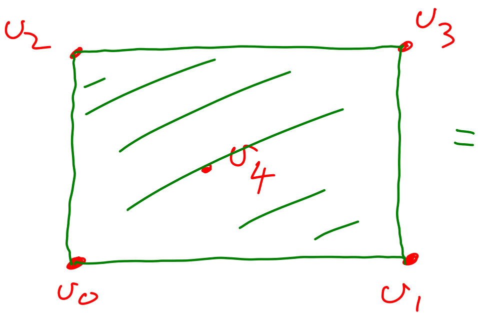
Let

$C = \mathrm{Conv}(\{v_0, v_1, \dots, v_k\})$

denote the "smallest" convex set in $\mathbb{R}^n$ containing $v_0, v_1, \dots, v_k$.

We call C the convex hull of $v_0, v_1, \dots, v_k$.

Example $v_0 = (0,0)$, $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (1,1)$, $v_4 = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2$.



$= \mathrm{Conv}(\{v_0, \dots, v_4\})$

Example

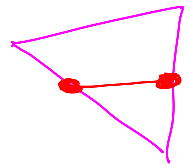
$v_0 = (1,0,0)$

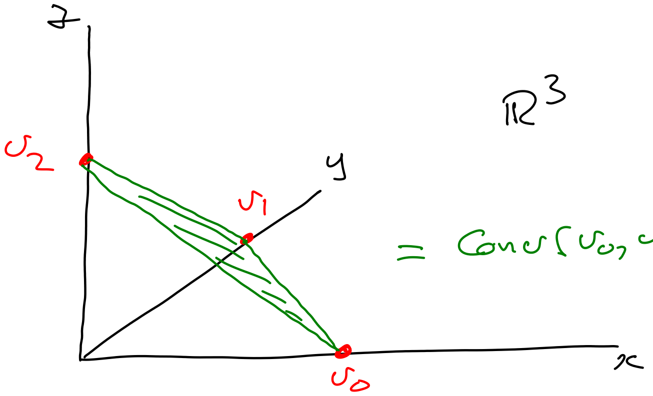
$v_1 = (0,1,0) \quad \in \mathbb{R}^3$

$v_2 = (0,0,1)$

$v_1 - v_0 = (-1,1,0)$

$v_2 - v_0 = (-1,0,1)$ are linearly independent





$\mathbb{R}^3$

$= \mathrm{Conv}(v_0, v_1, v_2)$

In general, for $v_0, v_1, \dots, v_k \in \mathbb{R}^n$,

$$C = \mathrm{Conv}(\{v_0, \dots, v_k\})$$

can be described as:

$$C = \left\{ \lambda_0 v_0 + \lambda_1 v_1 + \dots + \lambda_k v_k \ : \ \begin{array}{l} \lambda_i \in \mathbb{R} \\ \lambda_i \geq 0 \\ \sum_{i=0}^{k} \lambda_i = 1 \end{array} \right\}$$

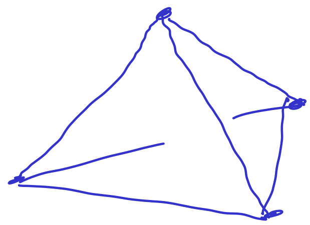
$\boxed{\begin{array}{l}\textbf{Defn} \quad \text{Let } v_0, v_1, \dots, v_k \in \mathbb{R}^n \text{ be vectors such that the } k \text{ vectors} \\ \qquad v_1 - v_0,\ v_2 - v_0,\ \dots,\ v_k - v_0 \\ \text{are linearly independent. We call} \\ \qquad C = \mathrm{Conv}(\{v_0, v_1, \dots, v_k\}) \\ \text{a } \underline{\text{simplex}} \text{ of dimension } k, \text{ or } k\text{-simplex.}\end{array}}$

0- simplex = point

1- simplex = line segment

2- simplex = Solid triangle

3- simplex = Solid tetrahedron



Simplices have "faces".
If A and B are simplices, and if the vertices of A form a subset of the vertices of B, then we say that A is a face of B.

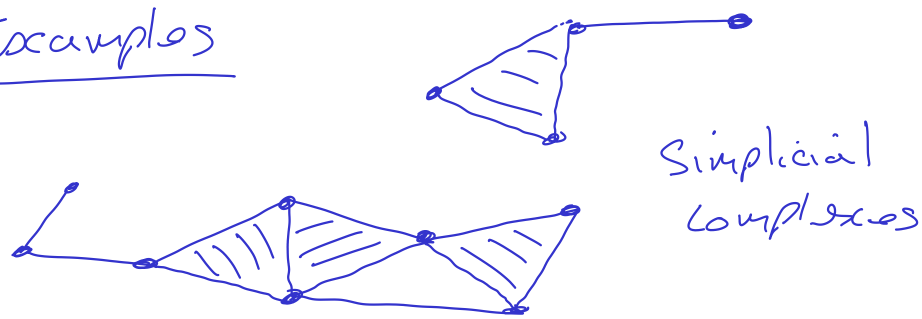
Example A 3- simplex has
- 4 faces of dimension 2
- 6 faces of dimension 1
- 4 faces of dimension 0
- 1 faces of dimension 3

Defn A finite collection K of simplices in $\mathbb{R}^4$ is called a simplicial complex if

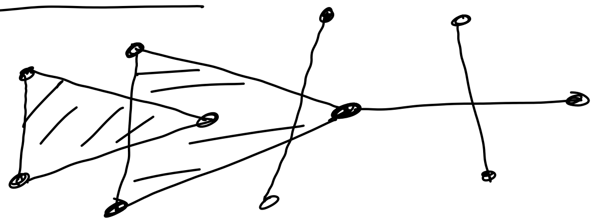
1) whenever a simplex lies in the collection then so too do all its faces,

2) whenever two simplices in the collection K intersect, they do so in a common face lying in K.

Examples



Non example



Any simplicial complex K is a subset of $\mathbb{R}^n$, and so it is a topological space with the (Euclidean) subspace topology.

We let $K, L \ldots$ denote simplicial complexes.

We let $|K|, |L|, \ldots$ to denote the corresponding topological spaces.

Defn Let X be a topological space. A triangulation of X consists of a simplicial complex K and a homeomorphism

$$h : |K| \longrightarrow X.$$

We define the Euler characteristic of X to be

$$\chi(X) = \chi(K)$$
$$= \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 - \cdots$$

where
$\alpha_k = \#$ of k-simplices of K.