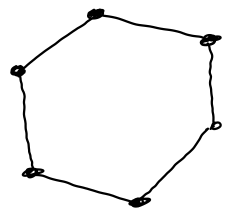
Defn A <u>triangulation</u> of a topological space X consists of a simplicial complex K and a homeomorphism

$$h : |K| \longrightarrow X .$$

Example Consider

$$X = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$$

Consider the simplicial complex

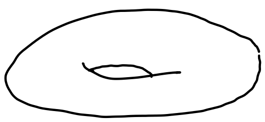
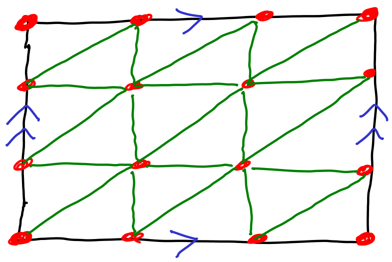


There is a homeomorphism

$$h : |K| \longrightarrow S^1 .$$

So this is an example of a triangulation of S^1.

Example Triangulation of the hollow torus

 $=$ 

$h : |K| \to S^1 \times S^1$

$S^1 \times S^1$

$\alpha_0 = 9$

$\alpha_1 = 27$

$\alpha_2 = 18$

This describes a simplicial complex homeomorphic to the torus.

$$\chi(\text{torus}) = 9 - 27 + 18 = 0$$

In general, for a simplicial complex K we let

α_0 denote the number of vertices of K

α_1 " " " " edges " "

α_k " " " " k-simplices " "

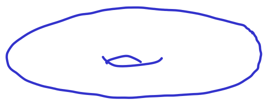
We define the Euler characteristic of a simplicial complex K to be

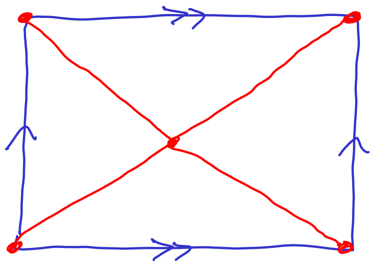
$$\chi(K) = \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 - \cdots$$

Defn Given a topological space X with a triangulation $h : |K| \to X$ we define the Euler characteristic of X to be

$$\chi(X) = \chi(K).$$

Example (Not a triangulation of the torus)



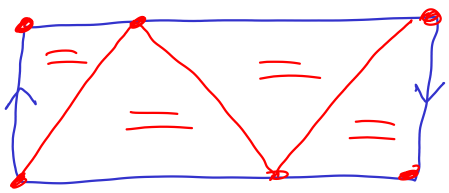


$\boxed{\chi(\text{torus}) = 2 - 6 + 4 = 0}$

There is no 1-simplex with just one vertex. There is no 2-simplex with just two vertices, so this is not a triangulation!

> **Theorem** (Most profound in the course)
>
> If two simplicial complexes K and L are such that $|K|$ is homeomorphic to $|L|$ then
>
> $$\chi(K) = \chi(L)$$

Example Determine the Euler characteristic of the Möbius band.



$\alpha_0 = 4$ $\qquad \chi(\text{Möbius band}) = 4 - 8 + 4 = 0$

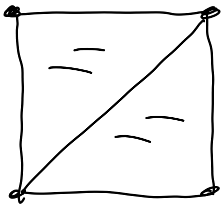
$\alpha_1 = 8$

$\alpha_2 = 4$

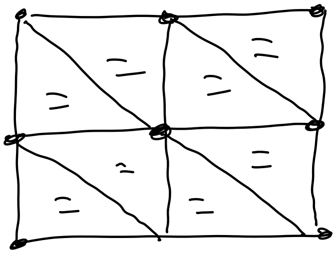
Example $\chi(S^2) = 2$.

Early attempts at proving the above theorem focused on:

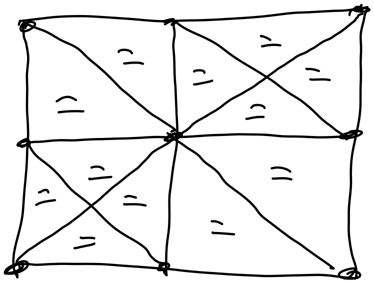
Hauptvermutung: If K and L are two triangulations of X then there are "subdivisions" K' of K and L' of L such that $K' = L'$.



K



L



$K' = L'$

The Hauptvermutung was proved for simplicial complexes of dimension ≤ 2. In 1961 John Milnor proved that the Hauptvermutung is not true for some simplicial complexes of dimension ≥ 3.