"Homotopy" is the key ingredient used in the proof of the topological invariance of Euler characteristic.

Defn Two maps $f: X \to Y$ and $g: X \to Y$ are <u>homotopic</u> if there exists a map

$$H: X \times [0,1] \longrightarrow Y, \quad (x, t) \longmapsto H_t(x)$$

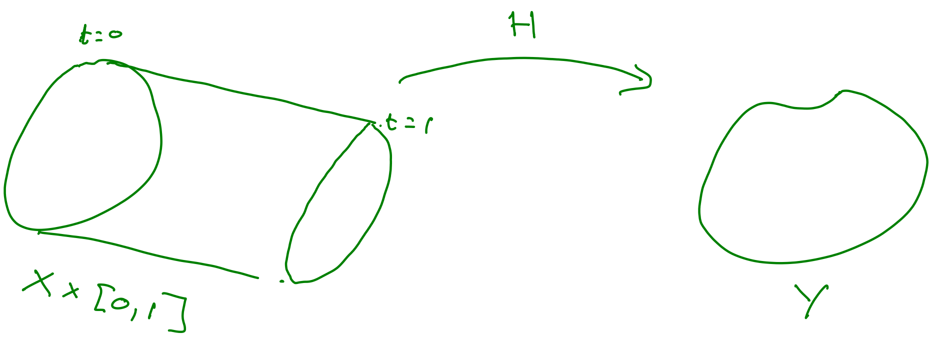
such that

$$H_0(x) = f(x),$$

and

$$H_1(x) = g(x)$$

for all $x \in X$.



We can think of $H_t(x)$ as a family of maps

$$H_t(x): X \longrightarrow Y, \quad x \longmapsto H_t(x)$$

which vary continuously with the parameter t.

we refer to H as a homotopy,
and we write $f \simeq g$.

To understand homotopy rigorously we need to understand the topology on $X \times [0,1]$.

Let $\mathcal{B}$ be the collection of all sets

$$U \times J = \{(u,j) : u \in U,\ j \in J\}$$

with U any open subset of X and J any open subset of $[0,1]$. A set is open in $X \times [0,1]$ if it is a union of sets in $\mathcal{B}$. [$\mathcal{B}$ is called a basis for the topology.]

Intuitively

$$H : X \times [0,1] \longrightarrow Y$$

is continuous if a small change in x and a small change in t yields only a small change in $H(x,t) = H_t(x)$.

Example Let $Y \subseteq \mathbb{R}^2$ be a convex subset of $\mathbb{R}^2$. Let X be <u>any</u> topological space.

<u>Any</u> two maps $f : X \to Y$, $g : X \to Y$ are homotopic. To see why we define a homotopy

$$H : X \times [0,1] \longrightarrow Y,$$

$$(x, t) \longmapsto f(x) + t\,(g(x) - f(x))$$

$$= (1-t) f(x) + t\, g(x)$$

describes the line from $f(x)$ to $g(x)$.

Note that H is continuous.

Note $H(x,t) \in Y$ for all $x \in X$, $t \in [0,1]$ because Y is convex.

Also $H(x,0) = f(x)$,

$H(x,1) = g(x)$.

Thus $f \simeq g$.

Proposition For fixed spaces X and Y, homotopy is an equivalence relation on the collection of all maps from X to Y.

proof

For any $f: X \to Y$ we have $f \simeq f$ (reflexive) thanks to the homotopy

$$H_t(x) = f(x).$$

For any $f: X \to Y$, $g: X \to Y$ if $f \simeq g$ then $g \simeq f$ (symmetric) because we can use the homotopy $H_t(x)$ for $f \simeq g$, with $H_0(x) = f(x)$ and $H_1(x) = g(x)$ to define a homotopy

$$H'_t(x) = H_{1-t}(x)$$

for $g \simeq f$.

For transitivity let $f, g, h : X \to Y$ be maps such that $f \simeq g$ and $g \simeq h$. So we have homotopies

$$H_t(x), \quad H_0(x) = f(x), \quad H_1(x) = g(x)$$

$$H'_t(x), \quad H'_0(x) = g(x), \quad H'_1(x) = h(x).$$

To see that $f \simeq h$ we define

$$H''_t(x) = \begin{cases} H_{2t}(x), & 0 \le t \le \frac{1}{2} \\ H'_{2t-1}(x), & \frac{1}{2} \le t \le 1 \end{cases}$$

with $H''_0(x) = f(x)$, $H''_1(x) = h(x)$. $\boxed{\text{ay}}$