Recall

$$D^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$$

$$\boxed{\begin{array}{l}\underline{\text{Brouwer's Theorem}} \\ \text{For any continuous function} \\ \qquad f : D^n \longrightarrow D^n \\ \text{there exists at least one point} \\ x \in D^n \text{ such that } f(x) = x.\end{array}}$$

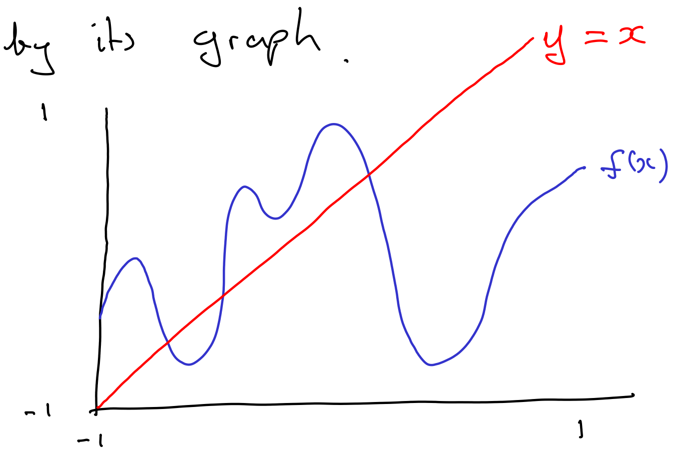
<u>Defn</u> If $f(x) = x$ we say that x is a <u>fixed point</u> of f.

<u>Case $n=1$</u> $D^1 = [-1, 1]$

we picture a map

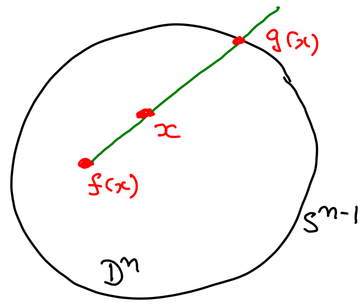
$$f : [-1, 1] \to [-1, 1]$$

by its graph.



A fixed point is a point where the blue graph of $f(x)$ intersects the red line $y=x$. Such a point exists by the Intermediate Value Theorem.

Proof of Brouwer's Theorem



Let $f: D^n \to D^n$ be continuous.

Suppose f has no fixed point.

Then we could define a continuous map

$$g: D^n \longrightarrow S^{n-1}, \quad x \longmapsto g(x)$$

where $g(x)$ is the point in S^{n-1} where the ray from $f(x)$ through x intersects the boundary.

Note that $g(x)$ is continuous.

Let $h: S^{n-1} \longrightarrow D^n$, $x \longmapsto x$

Now

$$gh: S^{n-1} \longrightarrow S^{n-1}, \quad x \longmapsto g(h(x)) = x$$

is clearly the identity on S^{n-1}.

Now $hg: D^n \longrightarrow D^n$ is homotopic to the identity on D^n, because of the homotopy

$$H: D^n \times [0,1] \to D^n,$$
$$H(x,t) = x + t(g(x) - x).$$

$H(x,0) = x$

$H(x,1) = g(x) = hg(x)$.

So $hg \simeq 1_{D^n}$

$gh \simeq 1_{S^{n-1}}$

Therefore D^n is homotopy equivalent to S^{n-1}.

Then our major theorem implies

$$\chi(D^n) = \chi(S^{n-1}).$$

But

$$D^n \simeq \{0\}$$

and so $\chi(D^n) = \chi(\{0\}) = 1$.

Last lecture:

$$\chi(S^{n-1}) = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}$$

Contradiction!

Hence f must have a fixed point. QED

Theorem (Frobenius - Perron)

Let A be an $n \times n$ matrix with real entries $a_{ij} > 0$.

Then A has a positive real eigenvalue. Moreover there is a corresponding eigenvector $v = (x_1, x_2, \dots, x_n)$, with $x_i \geq 0$ for all i.

Proof Define

$$\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad (x_1, \dots, x_n) \longmapsto \sum_{i=1}^{n} x_i$$

$$\Delta^{n-1} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}$$

Define

$$g : \Delta^{n-1} \longrightarrow \Delta^{n-1}, \quad x \longmapsto \frac{1}{\sigma(Ax)} Ax$$

Now g is continuous, and Δ^{n-1} is (homeomorphic to) D^{n-1}.

So Brouwer's Theorem implies that g has at least one fixed point,

$$x = g(x) = \frac{1}{\sigma(Ax)} Ax$$

So

$$\boxed{Ax = \sigma(Ax) \cdot x}$$

Therefore x is an eigenvector of A with eigenvalue $\lambda = \sigma(Ax) > 0$.

QED