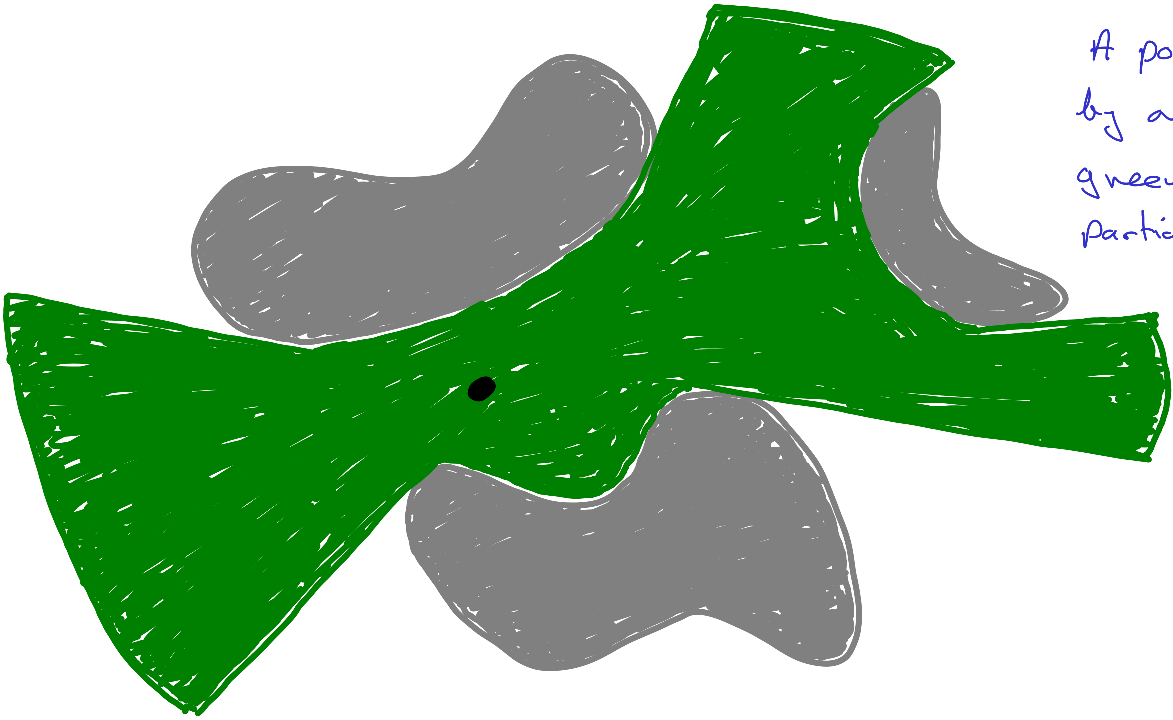
A Second Problem

A Texas farmer invests in thousands of cheap sensors, and spreads them all over the ranch. When activated from the farmhouse a sensor counts the number of cows within a certain distance r say, and within line of sight. The number of cows, and the sensor id is returned to the farmhouse.

How can the farmer determine the number of cows on the ranch from this data?



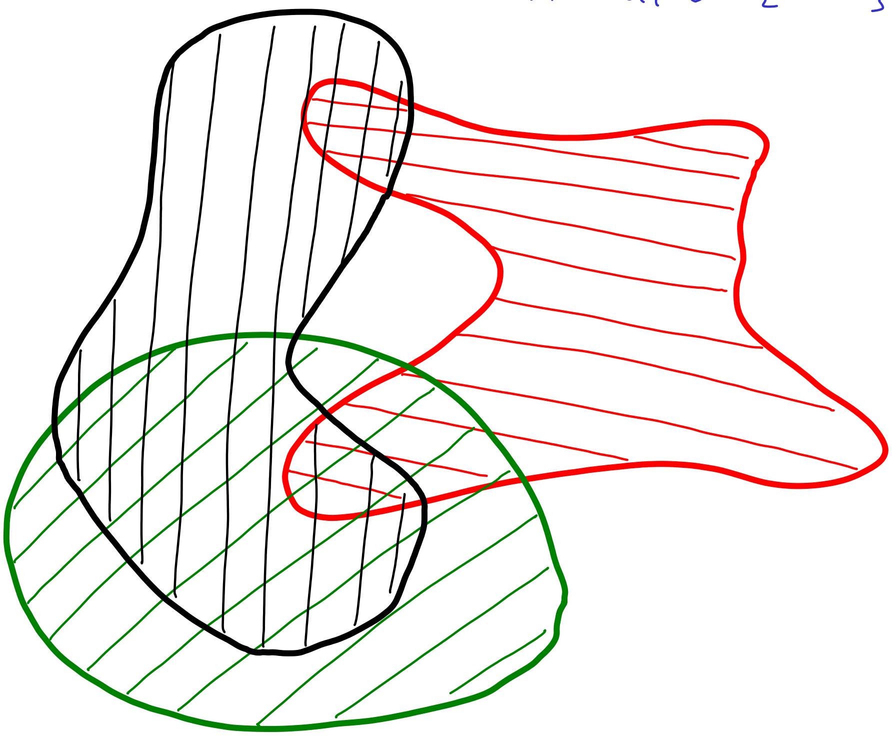
A point cow is detected by all sensors in the green area; it is partially hidden by the grey objects.

$X = U_1 \cup U_2 \cup U_3$ - entire r ch

U_1 = visibility region of cow 1

U_2 = visibility region of cow 2

U_3 = visibility region of cow 3



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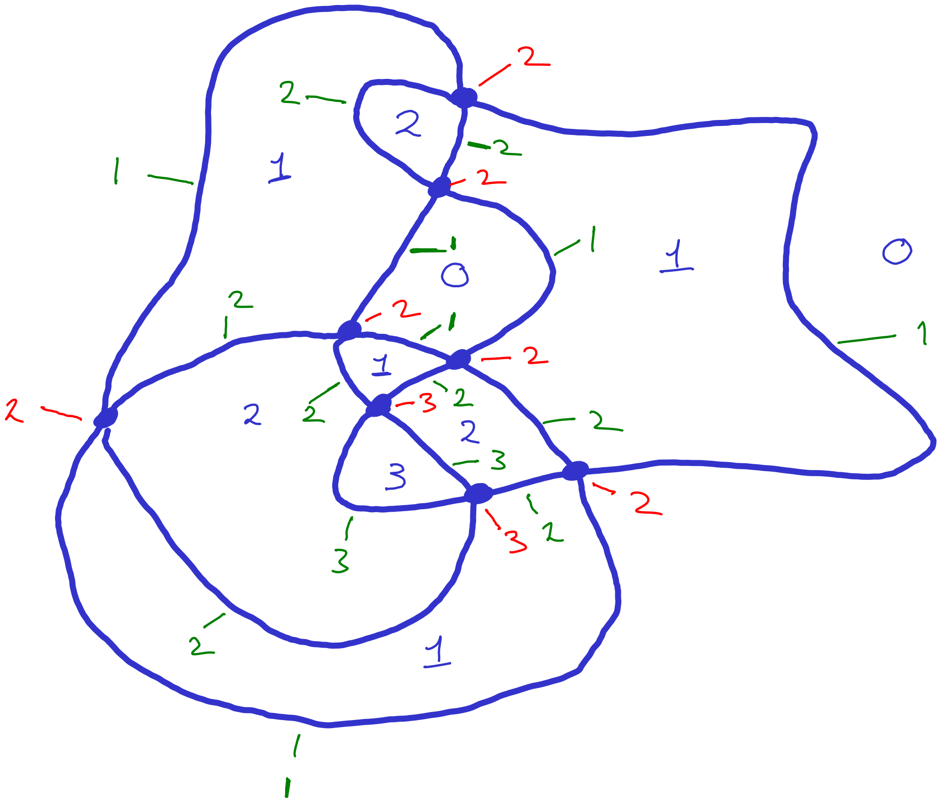

Given such a union

$$X = U_1 \cup U_2 \cup \cdots \cup U_t$$

we say that a function

$$\omega : X \longrightarrow \mathbb{Z}$$

is a weight function if it is constant on any given vertex, on any given edge, and on any given face.

Example The function

$$\omega(x) = |\{ i : x \in U_i \}|$$

(= number of colours used to colour x)

<u>Defn</u> Given a weight function

$$\omega : X = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \cdots \cup \mathcal{U}_t \longrightarrow \mathbb{Z}$$

we define the <u>Euler</u> <u>integral</u>

$$\int_X \omega \, d\chi = \sum_v \omega(v) - \sum_e \omega(e) + \sum_f \omega(f)$$

where v, e, f range over vertices, edges, faces and where $\omega(f)$ means $\omega(x)$ where $x \in f$.

Example (Texas ranch)

$$w(x) = |\{ i : x \in U_i \}|$$

Then

$$\int_X w \, dx = (6 \times 2 + 2 \times 3) - (6 \times 1 + 8 \times 2 + 2 \times 3) + (4 \times 1 + 3 \times 2 + 3)$$

$$= 18 - 28 + 13$$

$$= 3.$$

Theorem Let $X \subseteq \mathbb{R}^2$ be a region with subregions $U_1, U_2, U_3, \dots, U_t \subseteq \mathbb{R}^2$ such that

$$X = U_1 \cup U_2 \cup \cdots \cup U_t .$$

Let $\omega(x) = |\{i : x \in U_i\}|$.

Suppose that each region U_i has the same Euler characteristic

$\chi(U_i) = v - e + f = C \neq 0$. Then

$$t = \frac{1}{C} \int_X \omega \, d\chi .$$

Proof Let $1_{U_i} : X \to \mathbb{Z}$ be the weigh function defined by

$$1_{U_i}(x) = \begin{cases} 1 & \text{if } x \in U_i \\ 0 & \text{if } x \notin U_i . \end{cases}$$

Then

$$\int_X \omega \, d\chi = \int_X \Big(\sum_{1 \le i \le t} 1_{U_i} \Big) \, d\chi \overset{\text{Think!}}{=} \sum_{1 \le i \le t} \Big(\int_X 1_{U_i} \, d\chi \Big)$$

$$= \sum_{1 \le i \le t} \chi(U_i)$$

$$= t\, C .$$

QED.