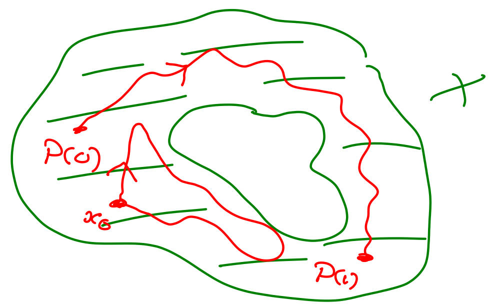
Third class test: 12 pm Wednesday 5 May

Topology and group theory are intimately related.

Let X be a topological space.



A continuous function

$$p : [0,1] \longrightarrow X$$

is called a path in X.

Choose $x_0 \in X$.

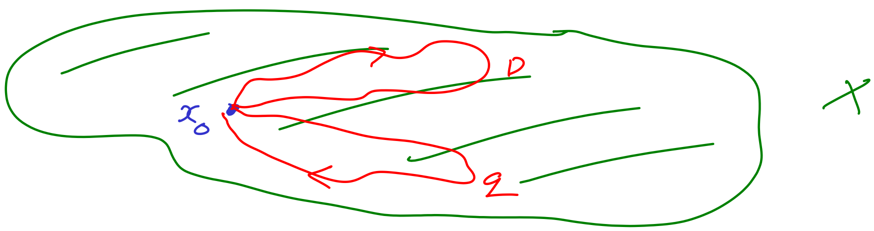
A path $p : [0,1] \to X$ with $p(0) = x_0$ and $p(1) = x_0$ is called a loops at x_0.

Given two loops $p, q : [0,1] \to X$ at x_0, we can combine them to form a new loop

$$p * q : [0,1] \longrightarrow X$$

at x_0 using the formula

$$p*q(t) = \begin{cases} p(2t), & 0 \le t \le \frac{1}{2} \\ q(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}.$$



This multiplication does not satisfy the axioms of a group. For instance, it doesn't admit inverses.

Two loops $p, q : [0,1] \to X$ at x_0 are said to be <u>homotopic</u> <u>rel</u> x_0 if there exists a continuous map

$$H : [0,1] \times [0,1] \to X, \quad (s,t) \mapsto H_t(s)$$

with

$$H_0(s) = p(s),$$

$$H_1(s) = q(s),$$

$$H_t(0) = x_0 = H_t(1) \quad \text{for all } t \in [0,1].$$

Homotopy rel x_0 is an equivalence relation of the set of loops at x_0.

Let $[p]$ denote the equivalence class of a loop p.

Let

$$\boxed{\pi_1(X, x_0) = \left\{ [p] : \begin{array}{l} \text{where } p : [0,1] \to X \\ \text{is a loop at } x_0 \end{array} \right\}}$$

<u>Theorem</u> (Henri Poincaré)

$\pi_1(X, x_0)$ is a group under the multiplication

$$[p] * [q] = [p * q].$$

<u>Proof</u> Not difficult. See Armstrong's book for details.

<u>Terminology</u> We call $\pi_1(X, x_0)$ the <u>fundamental</u> <u>group</u> of X at x_0.

Example $\quad S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

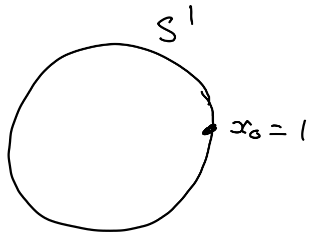
$1 \in S^1$

$\pi_1(S^1, 1) \cong \mathbb{Z}$ $\quad$ (= additive group of integers. $(\mathbb{Z}, +)$)
(also know as the infinite cyclic group)

See Armstrong for full details.

Idea

$X = S^1, \quad x_0 = 1$



$p_1 : [0,1] \to S^1, \quad \theta \mapsto p_1(\theta) = e^{2\pi i \theta}$

$p_2 := p_1 * p_1 : [0,1] \longrightarrow S^1, \quad \theta \longmapsto e^{4\pi i \theta}$

And

$(p_1 * p_1) * p_1$

is homotopic rel 1 to

$p_3 : [0,1] \longrightarrow S^1, \quad \theta \longmapsto e^{6\pi i \theta}$

In general, we have a loop

$p_n : [0,1] \to S^1, \quad \theta \mapsto e^{2n\pi i \theta}$

for each n.

One needs to show:

1) $[p_n] \neq [p_m]$ $\quad$ if $m \neq n$

2) Any loop
$q : [0,1] \longrightarrow S^1$
based at 1 is homotopic rel 1
to some p_n.

We say that q has winding number n.

3) $\quad [P_n * P_m] = [P_{n+m}]$.