Last lecture:

$$\pi_1(S^1, 1) \cong \mathbb{Z}$$

Fundamental Theorem of Algebra

Any polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $a_i \in \mathbb{C}$ and of degree $n > 0$ has at least one zero in $\mathbb{C}$.

Proof Since $a_n \neq 0$, a scalar multiply of the polynomial has the form

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

Let's suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$.

For $\lambda \geq 0$ define the map

$$f_\lambda : S^1 \longrightarrow S^1$$

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$

by

$$f_\lambda(z) = \frac{p(\lambda z)}{|p(\lambda z)|}.$$

Any two of these maps f_λ, $f_{\lambda'}$ are homotopic via the homotopy $H : S^1 \times [0,1] \to S^1$,

$$H_t(z) = \frac{p(((1-t)\lambda + t\lambda')z)}{|p(((1-t)\lambda + t\lambda')z)|}.$$

Note: For $\lambda = 0$ we have that f_0 is a constant function and has winding number 0.

Exercise (tricky)
For large λ we can check that $f_\lambda(z)$ is homotopic to

$$g_n : S^1 \to S^1, \quad z \mapsto z^n .$$

But g_n has winding number n.

But $g_n \simeq f_\lambda \simeq f_0$, and homotopic maps has the same winding number.

Hence the winding number of g_n is 0.

Contradiction: $n \neq 0$.

QED

Game Theory

A game involves

- n players
- a set S_i of strategies for player i
- a payoff function

$$v_i : S_1 \times S_2 \times S_3 \times \cdots \times S_n \longrightarrow \mathbb{R}$$

for each i, $i = 1, 2, \ldots, n$.

Example 1 Two players Mary and John. They want to go to the cinema (C) or to a soccer match (S) together.

$S_1 = \{C, S\}$, $S_2 = \{C, S\}$.

Mary $i=1$
John $i=2$

$v_1(C,C) = 2$ $v_2(C,C) = 1$	$v_1(C,S) = 0$ $v_2(C,S) = 0$
$v_1(S,C) = 0$ $v_2(S,C) = 0$	$v_1(S,S) = 1$ $v_2(S,S) = 2$

Example 2 Two players. Each places a coin on the table.

Player 1 wants coins to be the same.

Player 2 wants coins to be different.

$S_1 = \{H, T\}$, $S_2 = \{H, T\}$

Payoff function:

$v_1(H,H) = 1$ $v_2(H,H) = -1$	$v_1(H,T) = -1$ $v_2(H,T) = 1$
$v_1(T,H) = -1$ $v_2(T,H) = 1$	$v_1(T,T) = 1$ $v_2(T,T) = -1$

In a pure strategy game each player decides before the game on the strategy to play.

A pure Nash equilibrium occurs if, having played the game, no player benefits from unilaterally changing his/her choice of strategy.

Example 1 There are two pure Nash equilibria:

- both go to Cinema
- both go to Soccer

Example 2 There is no pure Nash equilibrium.