Recall A game involves

- n players
- a set S_i of strategies for player i
- a pay-off function

  $$v_i : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$$

  for each player i, $1 \le i \le n$.

Example $n = 2$ $S_1 = \{H, T\}$, $S_2 = \{H, T\}$

$v_1(H, H) = 1$ $v_2(H, H) = -1$	$v_1(H, T) = -1$ $v_2(H, T) = 1$
$v_1(T, H) = -1$ $v_2(T, H) = 2$	$v_1(T, T) = 1$ $v_2(T, T) = -1$

Defn A mixed strategy is a choice of probabilities

$$P_{i,s} = \text{probability that player } i \text{ plays strategy } s \in S_i$$

for $1 \le i \le n$, $s \in S_i$ satisfying

$$P_{i,s} \geq 0 , \quad \sum_{s \in S_i} P_{i,s} = 1 .$$

Notation Suppose $S_i = \{ s_1, s_2, \dots, s_k \}$

and

$$\mathbb{P}_i = (P_{i,s_1} , P_{i,s_2} , \dots , P_{i,s_k})$$

Define the expected payoff for player i to be the function

$$\mathcal{E}_i (\mathbb{P}_1 , \mathbb{P}_2 , \dots , \mathbb{P}_n) = E (v_i)$$

$$= \sum_{\substack{x_1 \in S_1 \\ x_2 \in S_2 \\ \vdots \\ x_n \in S_n}} P_{1 x_1} P_{2 x_2} P_{3 x_3} \cdots P_{n x_n} v_i (x_1, x_2, \dots, x_n) .$$

A mixed Nash equilibrium occurs if, having played the game, no player benefits from unilaterally changing their mixed strategy (the mixed strategies of all others remaining fixed).

Theorem (J. Nash) In any game with finitely many players and finite pure strategy sets, there exists at least one (mixed) Nash equilibrium.

Example for the above 2-player game

$S_1 = \{H, T\}, \quad S_2 = \{H, T\}$

$$E_i(P_1, P_2) = P_{1H} P_{2H} v_i(H,H) + P_{1T} P_{2H} v_i(T,H) + P_{1H} P_{2T} v_i(H,T) + P_{1T} P_{2T} v_i(T,T)$$

$$E_1 = P_{1H} P_{2H} - P_{1T} P_{2H} - P_{1H} P_{2T} + P_{1T} P_{2T}$$

$$E_2 = -E_1$$

In this 2-player game an example of a mixed Nash equilibrium is the mixed strategy

$P_{1H} = \frac{1}{2} \quad P_{1T} = \frac{1}{2} \quad P_{2H} = \frac{1}{2} \quad P_{2T} = \frac{1}{2}$.

Outline proof of Nash's Theorem

Consider

$$C = \{ (\underline{P}_1, \underline{P}_2, \underline{P}_3, \dots, \underline{P}_n) \} \subseteq \mathbb{R}^{|S_1|+|S_2|+\dots+|S_n|}$$

where $\underline{P}_i \in \mathbb{R}^{|S_i|}$ is the probability distribution for player i.

Now $P_{i,s} \geq 0$, $\sum_{s \in S_i} P_{i,s} = 1$ mean that C is closed, bounded and convex.

Thus, by Brouwer's Theorem, any continuous function

$$f : C \longrightarrow C$$

has at least one fixed point.

For a given $(\underline{P}_1, \underline{P}_2, \dots, \underline{P}_n) \in C$ define $\underline{q}_i \in \mathbb{R}^{|S_i|}$ to be "the" probability distribution that maximizes

$$E_i(\underline{P}_1, \underline{P}_2, \dots, \underline{P}_{i-1}, \underline{q}_i, \underline{P}_{i+1}, \dots, \underline{P}_n) \qquad (*)$$

Now define $f : C \longrightarrow C$ by

$$f(\underline{P}_1, \underline{P}_2, \dots, \underline{P}_n) = (\underline{q}_1, \underline{q}_2, \dots, \underline{q}_n) .$$

This function has a fixed point. But this fixed point is, by construction, a mixed Nash equilibrium. $\boxed{\text{My}}$

Slight problem: The quantity q_i that maximizes (*) may not be unique. Thus f is not a well-defined function.

To overcome this problem one replaces (*)

$$\Sigma_i(\underline{p}_1, \dots, \underline{p}_{i-1}, \underline{q}_i, \underline{p}_{i+1}, \dots, \underline{p}_n) - \|\underline{p}_i - \underline{q}_i\|^2 \qquad (*)$$