Defn (Riesz [1909], Hausdorff [1914])

A topological space consists of a set X and a collection T of subsets of X, each of which we call open. The following hold:

T1) The union of any collection of open sets is open.

T2) The intersection of any finite collection of open sets is open.

T3) Both ϕ and X are open

Example 1 $X = \{1, 2, 3, 4\}$

$T = \{\phi, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}$

This is a topological space.

Example 2 $X = \{1, 2, 3, 4\}$

$T = \{\phi, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\}$.

Not a topological space because T2 fails.

Example 3 $X = \{1, 2, 3, 4\}$

$T = \{\phi, \{2\}, \{3\}, \{1,2,3\}, \{1,2,3,4\}\}$.

Not a topological space because T1 fails.

Example 4 $X = \{1, 2, 3, 4\}$

$T = \{\phi, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}$

This is a topological space.

Example 5 Let $X = \mathbb{Z}$. The cofinite topology on X has, as open sets, those subsets $U \subseteq X$ such that the complement $X \setminus U$ is finite; also we deem $\emptyset$ to be open. This is an example of a topological space.

For the next example we need some notation.

For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ we define the Euclidean norm

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

For $x, y \in \mathbb{R}^n$ we define the Euclidean distance

$$d(x,y) = \|x-y\|.$$

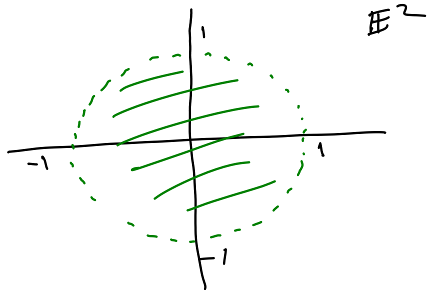
We write $\mathbb{E}^n$ to denote the set $\mathbb{R}^n$ endowed with this Euclidean distance.

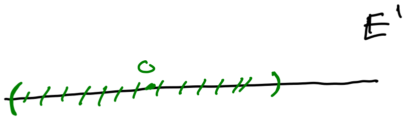
For $x \in \mathbb{E}^n$ and for any real number $\varepsilon > 0$ we define the open ball at x of radius ε to be

$$B^n(x, \varepsilon) := \{ y \in \mathbb{E}^n : d(x,y) < \varepsilon \}$$

Examples

$B^2((0,0),\ 1)$





$B^1(0,\ 1)$

$= (-1,\ 1)$

Example 6 $\quad X = \mathbb{R}^n$

Let T consist of those subsets $U \subseteq \mathbb{R}^n$ such that, for any $x \in U$ we can find $\varepsilon > 0$ such that the Euclidean ball

$$B^n(x,\ \varepsilon)$$

lies entirely with U. i.e.

$$B^n(x,\ \varepsilon) \subseteq U$$

This defines a topological space, which we denote by $\mathbb{E}^n$.

Defn A topological space X is said to be connected if it can not be expressed as a union

$$X = U \cup V$$

where U, V are non-empty open subsets of X with $U \cap V = \phi$.

Example 1 is connected.

Example 4 is connected

Example 5 $\mathbb{Z}$ is connected with the cofinite topology.

Example 6 $\mathbb{E}^n$ is connected.

Example 7 Let $X = \mathbb{R}^n$. Let $\mathcal{T}$ consist of all subsets of X. Then $(X, \mathcal{T})$ is a topological space. We call $\mathcal{T}$ the discrete topology.

This space is not connected. For instance

$$U = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$$

$$V = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}.$$

Then $X = U \cup V$, $U \cap V = \phi$, $U \neq \phi \neq V$.