Example Let $X = \mathbb{R}^n$. Let $\mathcal{T}$ consist of just two subsets of X, namely

$$\mathcal{T} = \{ \phi , X \}.$$

Then $(X, \mathcal{T})$ is a topological space. We call $\mathcal{T}$ the trivial topology on X.

The space X is connected.

Defn Let X be a set with a topology $\mathcal{T}$. Let $Y \subseteq X$ be a subset of X. In the subspace topology on Y a subset $U \subseteq Y$ is deemed to be open if and only if

$$U = Y \cap A$$

with A an open set in X.
With this topology on Y we say that Y is a topological subspace of X

Example $X = \mathbb{R}$ with the standard topology in which a subset $U \subseteq \mathbb{R}$ is open if, for any $x \in U$, there is an $\varepsilon > 0$ with

$$(x-\varepsilon, x+\varepsilon) \subseteq U.$$

Consider the integers $\mathbb{Z} \subseteq \mathbb{R}$, with the subspace topology.

Then $\mathbb{Z}$ is not connected because

$$U = \{ n \in \mathbb{Z} : n \leq 0 \}$$

$$V = \{ n \in \mathbb{Z} : n > 0 \}$$

Clearly $\mathbb{Z} = U \cup V$, $U \cap V = \emptyset$, and $U \neq \emptyset \neq V$ and U, V are open because

$$U = \mathbb{Z} \cap (-\infty, 1)$$

$$V = \mathbb{Z} \cap (\tfrac{1}{2}, \infty)$$

Definition A connected component of a topological space X is a connected subspace $Y \subseteq X$ such that there is no connected subspace $W \subseteq X$ with $Y \subsetneq W$.

Example Let $X = \{(x,y) \in \mathbb{E}^2 : x^2+y^2 \neq 1\}$

There are two connected components, namely

$$Y = \{(x,y) \in \mathbb{E}^2 : x^2+y^2 > 1\}$$

and

$$Z = \{(x,y) \in \mathbb{E}^2 : x^2+y^2 < 1\},$$

Example For the real line $\mathbb{R}$ with standard topology we have the subset $\mathbb{Q} \subseteq \mathbb{R}$ of rational numbers. With the subspace topology on $\mathbb{Q}$ we see that $\mathbb{Q}$ is not connected. For instance

$$A = \{x \in \mathbb{R} : x > \pi\} = (\pi, \infty)$$

$$B = \{x \in \mathbb{R} : x < \pi\} = (-\infty, \pi)$$

Then take $U = \mathbb{Q} \cap A$, $V = \mathbb{Q} \cap B$, so

$$\mathbb{Q} = (\mathbb{Q} \cap A) \cup (\mathbb{Q} \cap B),$$

and $\mathbb{Q} \cap A$ is open in $\mathbb{Q}$

and $\mathbb{Q} \cap B$ is open in $\mathbb{Q}$

and $(\mathbb{Q} \cap A) \cap (\mathbb{Q} \cap B) = \phi$

and $\mathbb{Q} \cap A \neq \phi \neq \mathbb{Q} \cap B$.

Continuity

Defn Let X, Y be topological spaces.

A function

$$f : X \longrightarrow Y$$

is continuous if the inverse image $f^{-1}(U) = \{x \in X : f(x) \in U\}$ of any open set $U \subset Y$ is open in X.

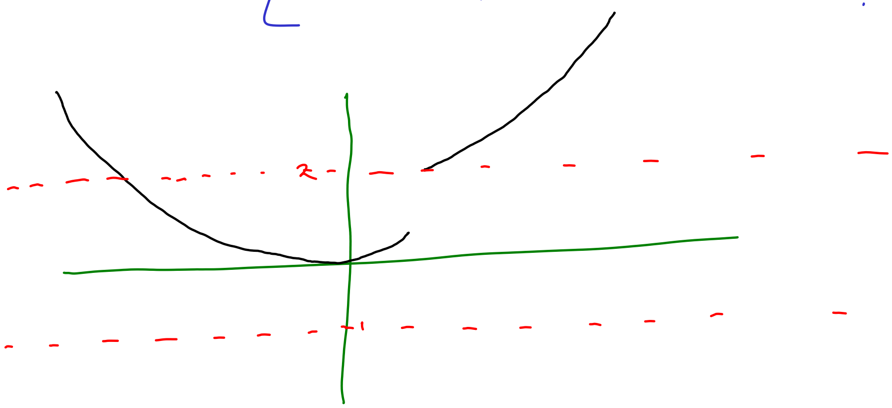
Example $X = \mathbb{E}^1$, $Y = \mathbb{E}^1$.

Consider $f : \mathbb{E}^1 \to \mathbb{E}^1$ given by

$X = \mathbb{E}^1$

$Y = \mathbb{E}^1$

$$f(x) = \begin{cases} x^2, & x \le 1 \\ x^2 + 1, & x > 1 \end{cases}$$



Consider

$U = (-1, 2) \subseteq \mathbb{E}^1 = Y$

The pre-image

$f^{-1}(U) = (-\sqrt{2}, 1]$.

Since the pre-image of $U = (-1, 2)$ is not open, we see that f is not continuous according to the above definition.