First test: Wednesday 10 March. 5 questions related to the homework sheet sections 1-6.

Example The spaces

$(-1, 1)$ and $\mathbb{R}$

are homeomorphic because of the homeomorphism

$$f : (-1,1) \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{x}{1-x^2}$$

Let's aim towards showing that the following spaces are distinct (not homeomorphic).

- $[-1, 1]$
- $\mathbb{R}$
- $\mathbb{R}^2$
- S^1

Recall A property is topological if, whenever X has the property, then so too does every space Y which is homeomorphic to X.

Proposition Let $f : X \to Y$ be a homeomorphism. If X is connected then so too is Y.

Proof Let's suppose that Y is not connected. Then there exist open sets $U, V \subset Y$ such that $Y = U \cup V$, $U \cap V = \phi$, $U \neq \phi \neq V$. Suppose $f : X \to Y$ is a homeomorphism. Since f is continuous

$$f^{-1}(U) = \{x \in X : f(x) \in U\}$$

is open in X. So too is $f^{-1}(V)$.

Since f is a homeomorphism (and hence surjective) we have

$$f^{-1}(U) \cup f^{-1}(V) = X.$$

Also

$$f^{-1}(U) \cap f^{-1}(V) = \phi.$$

and $f^{-1}(U) \neq \phi \neq f^{-1}(V)$.

Hence X is <u>not</u> connected.

$\boxed{\text{Ue}d}$

<u>Example</u> Let's show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$ (with standard topologies).

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ were some homeomorphism.

$$X = \mathbb{R}^2 \setminus \{(1,2)\}$$

$$Y = \mathbb{R} \setminus \{f(1,2)\}$$

<u>Exercise</u> If $f : \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism then so too is $f : X \to Y$.

But X is connected and Y is <u>not</u> connected.

Hence there is no homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}$.

Towards Compactness

We'd like to say that $[-1, 1]$ is "finite"

and that $\mathbb{R} = (-\infty, \infty)$ is "infinite".

However $(-1, 1)$ is homeomorphic (bijective) and $(-\infty, \infty)$.

We'll use the words "compact" and "non compact"

instead of "finite" and "infinite".

Next lecture we'll define compactness

and show its a topological property.

In a subsequence lecture we'll show that $[-1, 1]$ is

compact. It will be easy to see that $\mathbb{R}$ is not compact.

Hence we'll have shown that $\mathbb{R}$

is not homeomorphic to $[-1, 1]$.