We want to say that $[0, 1]$ is "compact" but that $\mathbb{R}$ is "not compact".

Let $X$ be a topological space.
Let $\mathcal{F}$ be a family of open subsets of $X$ whose union equals $X$. We say that $\mathcal{F}$ is an open cover of $X$.

Example 1 Let $X = \mathbb{R}$ with usual topology.

Let
$$\mathcal{F} = \{ (n-2, n+2) \}_{n \in \mathbb{Z}} .$$
Then $\mathcal{F}$ is an open cover of $\mathbb{R}$.

Example 2 Let $X = [0, 1]$ with the subspace topology of $\mathbb{R}$.
$$\mathcal{F} = \{ [0, \tfrac{1}{3}), (\tfrac{1}{4}, \tfrac{3}{4}), (\tfrac{2}{3}, 1] \} .$$
So $\mathcal{F}$ is an open cover of $X$.

Let $\mathcal{F}$ be some open cover of $X$.
If $\mathcal{F}'$ is a subfamily of $\mathcal{F}$ and if the union of all sets in $\mathcal{F}'$ is equal to $X$, then we say that $\mathcal{F}'$ is a subcover of $\mathcal{F}$.

<u>Example 3</u> $X = \mathbb{R}$

$\mathcal{F} = \{(n-2, n+2)\}_{n \in \mathbb{Z}}$

$\mathcal{F}' = \{(n-2, n+2)\}_{n \in 2\mathbb{Z}}$

Then $\mathcal{F}'$ is a subcover of $\mathcal{F}$ since the union of $\mathcal{F}'$ is <u>equal to</u> $X = \mathbb{R}$.

An open cover $\mathcal{F}$ of $X$ is said to be <u>finite</u> if $\mathcal{F}$ involves only finitely many sets.

$$\boxed{\text{\underline{Defn} A topological space } X \text{ is \underline{compact} if every open cover } \mathcal{F} \text{ of } X \text{ has a finite subcover.}}$$

<u>Example 4</u> $X = \mathbb{R}$ with usual topology.

Considering $\mathcal{F} = \{(n-2, n+2)\}_{n \in \mathbb{Z}}$

we see that $\mathbb{R}$ is <u>not</u> compact.

The following proposition states that compactness is a topological property.

<u>Proposition</u> Suppose that $f : X \to Y$ is a surjective continuous function. If $X$ is compact then so too is $Y$.

<u>Proof</u> Suppose $X$ is compact, and that $f$ is a surjective continuous function.

Let $\mathcal{F}$ be some open cover of $Y$. Then

$$\mathcal{G} = \{f^{-1}(U)\}_{U \in \mathcal{F}}$$

is an open cover of $X$ (by surjectivity of $f$).

Since $X$ is compact $\mathcal{G}$ must have some finite subcover $\mathcal{G}'$. Let's say

$$\mathcal{G}' = \{f^{-1}(U)\}_{U \in \mathcal{F}'}$$

with $\mathcal{F}'$ a finite subcollection of $\mathcal{F}$.

But

$$\{U\}_{U \in \mathcal{F}'}$$

is a finite subcollection of $\mathcal{F}$. Moreover $\{U\}_{U \in \mathcal{F}'}$ has union $Y$. Thus $\mathcal{F}'$ is a finite subcover of $Y$.

$\boxed{\text{W}}$

<u>Example 5</u> Consider $X = (0,1)$ with the usual topology on $\mathbb{R}$. The space $X$ is homeomorphic to $\mathbb{R}$ (last lecture). Therefore $X = (0,1)$ is <u>not</u> compact.

Theorem $[0,1]$ is compact.

Proof See second Oblivion homework.

Hence $[0,1]$ is not homeomorphic to $\mathbb{R}$.

Class test: 12 pm Wednesday 10 March.
Based only on sections 1-6 of the problem sheet.