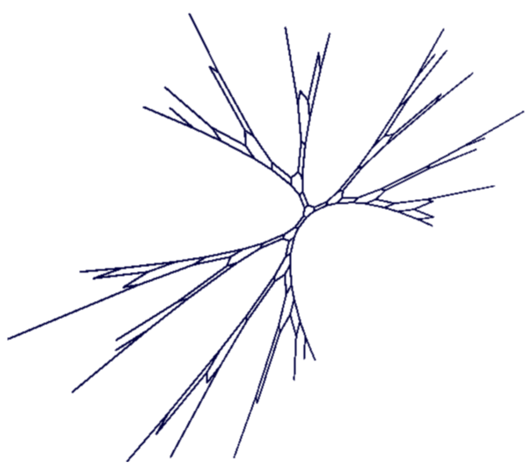
Topology

beyond a first syllabus



by NUIG undergraduates

2021

Topology project

Each MA342 student is required to contribute a co-authored article to a volume on *Topology: Beyond a First Syllabus*. The contribution counts for 25% of the module assessment.

Submission details

1. The submission deadline is 5pm on Friday 7 May 2021. The method of submission is still to be decided – but it will likely be as a Blackboard assignment.
2. Each article should be no more than five pages (2 authors) or eight pages (3 authors) maximum (including bibliography), should be written on a topology topic not covered in the MA342 exam syllabus, and should be written with a view to helping other MA342 students broaden their understanding of topology. The intended readership is MA342 students, and not professors of topology!
3. Each article should have at least two and no more than three co-authors.

   All co-authors will receive a common score for their contribution.
4. The article should be submitted as a pdf file produced using the Latex style file for the Proceedings of the American Mathematical Society. Please do not adjust the default page size and font size. The syle file is available in Overleaf.
5. You must follow all the AMS guidelines, and in particular:
   (a) Choose the most appropriate communicating AMS editor.
   (b) Include an abstract.
   (c) Include a primary Mathematical Subject Classification, and a secondary classification if appropriate.
   (d) Include a bibliography, with each bibliographic entry being cited at least once in the body of the article.

Possible topics

The MA342 exam syllabus is defined by the problem sheet, the lectures, and recent past exams. You are free to write on anything that complements, or expands on, this syllabus and that will help to improve other MA342 students' general understanding of the area of topology. You could choose an interesting definition and illustrate it and its use. You could choose a theorem and illustrate what it says. You could state a theorem and present a proof. You could write about some aspect of the life of a topologist. You could write about a recent trend in topology. The possibilities are endless. A few specific ideas are listed below.

- Discuss the quote:

  > Topology! The stratosphere of human thought! In the twenty-fourth century it might just possibly be of use to somebody, but for the present ...
  >
  > —— Solzhenitsyn, In the First Circle

- Explain/illustrate what the Poincaré Conjecture says, and write about the history of its proof. A statement of the conjecture, which is now a theorem, can be found on the Wikipedia topology page.

- State and prove any theorem from the later chapters of M.A. Armstrong's book *Basic Topology.*

- State and illustrate one of the topology theorems listed on the Wikipedia *Theorems in Topology* page.

- State and prove any theorem from Aisling McCluskey's book *Undergraduate Topology: A Working Textbook*

- What is a knot, what are the main goals of knot theory, and what were the origins of knot theory? (See for instance M.A. Armstrong's book *Basic Topology.*)

- Explain/illustrate what is meant by the homology of a simplicial complex. How does it relate to the Euler characteristic of a simplicial complex? (See for instance M.A. Armstrong's book *Basic Topology.*)

- Explain the Mapper clustering algorithm due to Singh, Mémoli, and Carlsson. Use a Google search to get started. If you are really ambitious you could even give an example using the R-package TDAmapper.

- Explain/illustrate what persistent homology is, and how it is used to understand point cloud data sets. Use a Google search to get started. If you are really ambitious you could even give an example using the R-package TDA.

- Write about topologists and the Fields medal. Which topologists have won the medal? Why did they win it? To get started you could look at the Wikipedia page on the Fields Medal.

- And so forth ...

THE INSCRIBED RECTANGLE OF A SMOOTH JORDAN CURVE

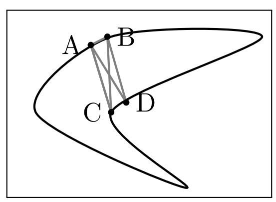
BYRNE,CATHAL, HICKEY,AMY, AND KANE,BAILEY

Abstract. The basis of this article is to show using topological invariants that for any continuous simple closed curve (Jordan curve) there exist four points of the curve that form the vertices of a rectangle and explore some properties of said rectangle.

[1]

1. Rectangle Peg Problem

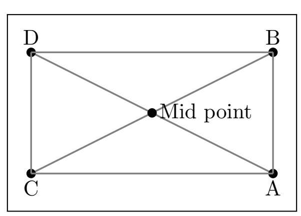
The inscribed rectangle problem is derived from the Topelitz problem which asks whether any Jordan curve contains four vertices of a square. While it has been proven true for any simple Jordan curve satisfying a high degree of smoothness, it remains unsolved for the general continuous case.



In order to form a rectangle using points on the curve Υ we must first recollect theorems pertaining to rectangles.

Theorem 1.1. *If a parallelogram has congruent diagonals, it is a rectangle.*

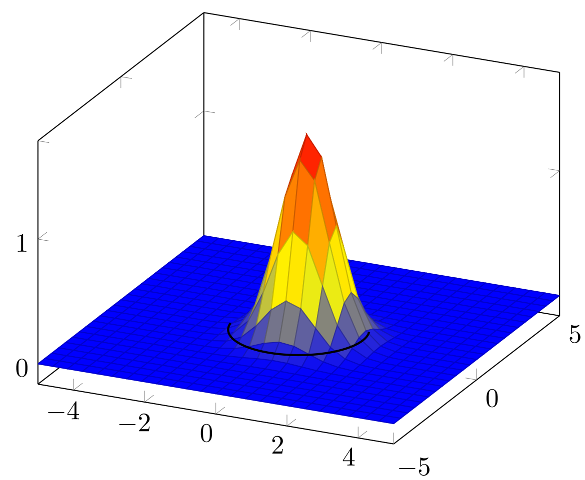
(1) *The diagonal distance between vertices of a rectangle are equal.*
(2) *The diagonal lines between vertices have the same midpoint.*



Received by the editors 4.
2010 *Mathematics Subject Classification.* Primary 57N16,57N65 ; Secondary 51H99,51H25.
[1]Editor: Shelly Harvey
Rice University Houston, TX 77251-1892 USA
shelly@rice.edu

This observation simplifies the problem greatly.In order for a rectangle to exist we must find two pairs of distinct point s.t that the distance between the point of the pairs are equal and that the pairs of point share the same midpoint.

A function(continuous map) $F(x_1, x_2)$ where $x \in \Upsilon$ can be used to represent this idea[3]. $F(x, y)$ takes in a pair of points on the curve and outputs a single point in three dimensional space.The curve can be considered to lie on a plane.$F(x_1, x_2)$ records the mid point $\dfrac{x_1 + x_2}{2}$ and distance $|x_1 - x_2|$ between pairs of points.$F(x_1, x_2)$ maps the input to the point $|x_1 - x_2|$ above (z direction) the midpoint of the pair.The function will produce a surface which hugs the curve.



Near the curve $F(x_1, x_2)$ will tend towards the curve as $F(x, y)$ takes pairs of points progressively closer to each other. along the curve the function will take in a pair of numbers that are the same outputting that same point.

$$F(x_1, x_1) = x_1$$

If two pairs of points have the same midpoint and distance $F(x_1, x_1) = F(x_3, x_4)$, meaning the surface intersects at that point, such an intersection creates the necessary conditions for a rectangle.

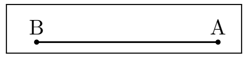
To further explore this problem we must consider how to present the pairs of points on the curve.The topological component of this article stems from finding certain 2 d surfaces to represent the entire collection of points on the curve.The properties of these surface will eventually detail how and why the surface created by $F(x_1, x_2)$ must intersect. When considering pairs of points we can class pairs as ordered

$$(A, B) \neq \ (A, B)$$

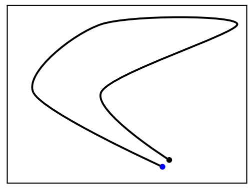
Or unordered

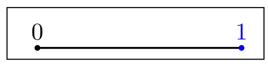
$$(A, B) = (A, B)$$

It is clear to see that we must focus on unordered pairs as ordered pairs leads to the possibility of finding two distinct pairs (A, B) and (B, A) which have the same midpoint and distance apart ie(intersect), but clearly do not create a rectangle



However for the moment lets entertain the idea. A first step is to cut the curve at a point and deform it into an interval. The interval should correspond to the curve with each point on the curve having a complementary point on the interval.

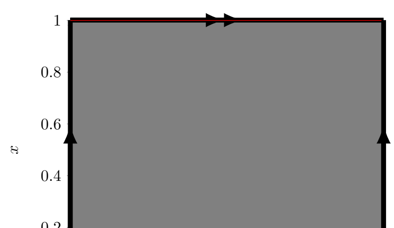




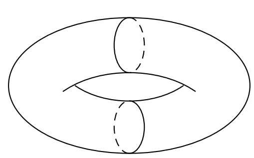
The point at which the curve is cut is simultaneously represented by the start and end of the interval.If the interval is copied and lined up such that the two intervals form an axis.We can now approach our pairs of points similar to how pairs of points in $\mathbb{R}^2$ are presented.The resulting graph can be filled to create a $1 * 1$ square.Every point in this square correlates to a pair of points on the curve.A complication arises from the fact 0 and 1 represent the same point on the curve.In order to create an accurate map of the curve these points must be glued back.

For all points where $x = 0$ and $0 < y \leq 1$ there is a complimentary point where $x = 1$ and $0 < y \leq 1$ such that both represent the same point on the curve.The left and right side of the square must be glued.

Similarly for all points where $y = 0$ and $0 < x \leq 1$ there is a complimentary point where $y = 1$ and $0 < x \leq 1$ such that both represent the same point on the curve.



If the 2d surface was glued accordingly the resulting shape is a torus.



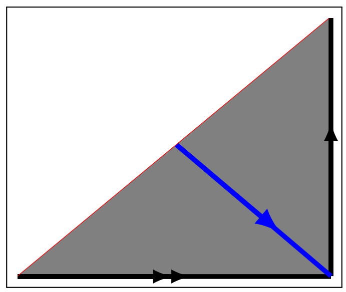
In fact the torus represents pairs of ordered points on the curve.

In order to represent the unordered pairs of points on the curve alterations must be made to the square.Points on the square which are considered the same when we class all pairs as unordered are glued together similar to our previous alteration of the square.

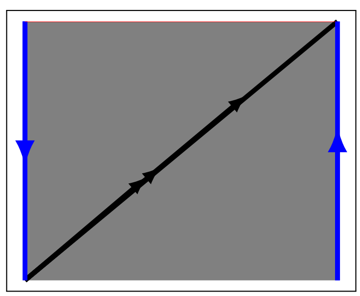
$$(0.8, 0.5) = (0.5, 0.8)$$

A diagonal line is created along which the square is folded onto its self creating a right angled triangle. This diagonal line represents each pair of points consisting of the same point on the curve i.e. (x, x) e.g $(0.7, 0.7)$.

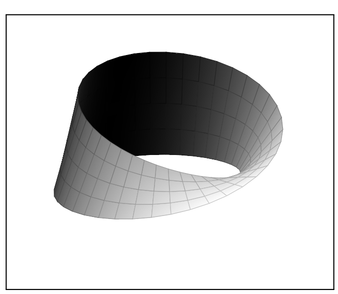
The remaining sides must be glued together , to achieve this the triangle is cut along the line bisecting the 90° angle.



Once cut the separated parts are reattached according to the original orientations of the triangle.



A mobius strip is created from the 2d surface obtained.The mobius strip is a representation of the unordered pairs of points on the closed curve.The edge of the mobius strip corresponds to the pairs of points consisting of the same point.



As the original function $F(x_1, x_2)$ is homeomorphic to the Mobius strip, the surface created by $F(x_1, x_2)$ can be represented by the mobius strip.The surface of

$F(x_1, x_2)$ hugged tightly to the curve as pairs of points tended to each other.There is a continuous association between unordered pairs of points in the curve and unique points on the strip.The mobius strip can be deformed to the surface of $F(x_1x_2)$ with the edge lying on the curve outlined in the plane. Due to the nature of the the mobius strip in order to form the surface of the function the strip must intersect i.e. the surface intersects.As the surface must intersect , there exists pairs of distinct points who share the same mid point and have the same distance between the points in the pair.The existence of these points implies the existence of 4 vertices of a rectangle in the curve.Therefore any curve contains four points representing the vertices of a rectangle.

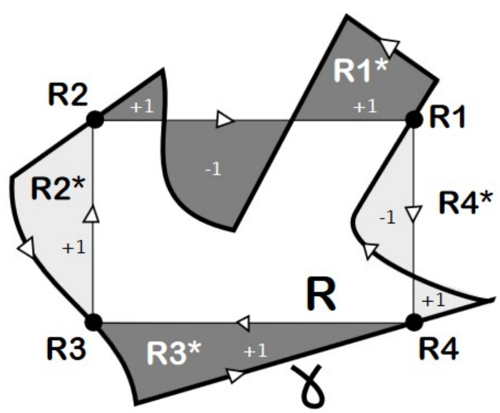
2. The Integral Formula

Now that we know The Inscribed Rectangle exists lets look at some interesting properties it has. We'll begin by looking at **The Integral Formula**.

Lets derive this formula.

Some definitions:

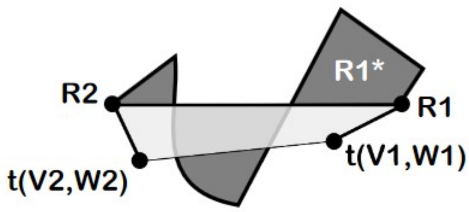
- Let γ be a counterclockwise orientated piecewise smooth Jordan curve.
- **R** denotes a rectangle inscribed in γ with vertices R_1, R_2, R_3, R_4 labelled counterclockwise.
- for $j \in \{1, 2, 3, 4\}$, A_j denotes the signed area of region, R_j^*, bounded by the segment $\overline{R_jR_{j+1}}$ and the arc of γ connecting $R_j \& R_{j+1}$. A_j lies in between R_jR_{j+1}, points in counterclockwise order.
- The signed area of R_j^* is determined from taking the integral of the winding number function over R_j. *Note:* if γ is convex all signed areas are positive.
- for fixed γ, $A(R) = (A_1 + A_3) - (A_2 + A_4)$
- $(X, Y) \in \mathbb{R}^2$ where $X = length(\overline{R_1, R_2})$ and $Y = length(\overline{R_2, R_3})$.
- $R(t)$ of rectangles of γ have $A(t) = A(R(t))$ where $(X(t), Y(t)) = (X(R(t)), Y(R(t)))$
- Shape curve, $Z(t) = (X(t), Y(t))$



e.g. labelled vertices & signed area of R_j^*

Lemma 2.1. $\frac{dA}{dt} = Y\frac{dX}{dt} - X\frac{dY}{dt}$

Proof. It is sufficient to prove for the case t=0. Rotate **R** such that the first side of **R(0)** is contained in an horizontal line, shown in figure below.



Evaluate all derivatives at t=0;
$\frac{dR_j}{dt} = (V_j, W_j)$
Up to 2nd order the region $R_1^*(t)$ is obtained by adding a small quadrilateral with base X(0) & adjacent sides parallel to $t(V_1, W_1)$ & $t(V_2, W_2)$.
The area of this quadrilateral is $\frac{tX(W_1+W_2)}{2} \implies$

$$\frac{dA_1}{dt} = -\frac{X(W_1 + W_2)}{2} \tag{2.1}$$

We have a negative sign because the area of the region increases when $W_1 \& W_2$ are negative. Similarly we find that

$$\frac{dA_3}{dt} = +\frac{X(W_3 + W_4)}{2} \tag{2.2}$$

Adding these we find that

$$\frac{dA_1}{dt} + \frac{dA_3}{dt} = X \times \frac{(W_3 - W_1)}{2} + X \times \frac{W_4 - W_2}{2} = -X[\frac{dY}{dt}] \tag{2.3}$$

Similarly we find

$$\frac{dA_2}{dt} + \frac{dA_4}{dt} = -Y[\frac{dX}{dt}] \tag{2.4}$$

So subtracting **2.4** from **2.5** we find that:

$$\frac{dA}{dt} = Y\frac{dX}{dt} - X\frac{dY}{dt} \tag{2.5}$$

as required. □

Now we let $w = \frac{dA}{dt}$. Integrating the result of the Lemma we get

$$A(1) - A(0) = \int_Z w \tag{2.6}$$

Letting 0=(0,0), consider the closed loop :
Z' $= \overline{0, Z_0} \cup Z \cup \overline{Z_1, 0}$.
Note that w vanishes on vectors of form (x,x) $\implies$
$A(1) - A(0) = \int_Z w = \int_{Z'} w =$

$$-\iint_\Omega 2\, dx\, dy = -2area(\Omega)$$

Where Ω is the region bounded by Z'.

So we have found an Integral formula for counterclockwise labelling of the vertices of **R**. Lets now consider a model free from labelling.

Let A_γ be the area of the region bounded by γ. When 2 sides of $\mathbf{R}$ are very close together $A_i + A_{i+2}$ is very near $\pm A_\gamma$. The equation is exact in degenerate limit.
When one side of $\mathbf{R}$ is very short, the corresponding A_i is either close to 0 or $\pm A_\gamma$. Therefore when aspect ratio of $\mathbf{R}$ is near 0 or ∞ , $(A_1 + A_3) - (A_2 + A_4)$ is very close to $kA_\gamma, k \in \mathbb{Z}, |k| \leq 3$. The result is exact is degenerate limit.
Take a path of rectangles $t \mapsto R(t), t \in [0,1]$ which starts & ends with rectangles of aspect ratio 0, then $K(0) \& K(1) \in \mathbb{Z}$ corresponding to each end of the path coincide. Thus, the region Ω corresponding to this region has signed area 0. The same applies if aspect ratios $\longrightarrow \infty$ at either end of the path.

3. A Theorem

It can be proven that every smooth Jordan curve has an inscribed rectangle of aspect ratio $\sqrt{3}$

Theorem 3.1. *Let $\gamma : S^1 \mapsto \mathbb{C}$ be a C^∞ injective function with nowhere vanishing derivative. Then $\forall n \geq 2 \; \exists \; k \in \{1, \ldots, n\text{-}1\} \in \mathbb{Z}$ such that γ has an inscribed rectangle of aspect ratio $= tan(\frac{\pi k}{2n})$*

A consequence of **Theorem 3.1** is the following:

Corollary 3.1.1. Every smooth Jordan curve has an inscribed rectangle with an aspect ratio of $\sqrt{3}$

Before we prove this, we must first consider the Lemma:

Lemma 3.2. *Let K_n denote the knot in $\mathbb{C} \times S^1$ parameterized by $g \mapsto (g, g^{2n})$ for $g \in S^1$. Then if $n \geq 3$ there is no smooth curve embedding of the Mobius strip $M \mapsto \mathbb{C} \times S^1 \times \mathbb{R}_{\geq 0}$ such that $\delta M \mapsto K_n \times \{0\}$*

Proof. The manifold $\mathbb{C} \times S^1$ is isomorphic to the manifold of the interior of the solid torus. The image of K_n under any embedding of the solid torus into S^3 yields a knot in S^3 which must have a non-orientable 4-genus at most that of K_n. By embedding the solid torus with a single axial twist the image of K_n becomes the torus knot $T_{2n}, n-1$. *Joshua Baston* used *Heegard-FLoer* homology to show non-orientable 4-genus of the torus knot $T_{2n}, n-1$ is at least n-1.

Therefore for $n \geq 3$ the knot K_n can not be bounded by a Mobius strip in the 4-Manifold $\mathbb{C} \times S^1 \times \mathbb{R}_{\geq 0}$ □

We will now use this fact to prove **Theorem 3.1**

Proof. Theorem 3.1:
Let $\gamma : S^1 \mapsto \mathbb{C}$ be a C^∞ injective function with a nowhere vanishing derivative. Assume there is not an inscribed rectangle with aspect ratio $tan(\frac{\pi k}{2n})$ for $k \in \{1, ..., n-1\}$. By the smooth inscribed square theorem we know $n \geq 3$
For Mobius strip, M, define $\mu : M \mapsto \mathbb{C}^2$ by

$$\mu(x,y) = (\frac{\gamma(x) + \gamma(y)}{2} \cdot (\gamma(y) - \gamma(x))^{2n}) \tag{3.1}$$

If $\mu(x,y) = \mu(w,z)$ then they have a shared midpoint, m and the angle at m must be a multiple of $\frac{\pi}{n}$. Therefore either {x,y} = {w,z} or (x, y, w, z) form the vertices of a rectangle with an aspect ratio of $tan(\frac{\pi k}{2n}), k \in \{1, ..., n-1\}$.

Furthermore, it is clear the differential of M is non-degenerate $\implies \mu$ is a smooth embedding of the Mobius strip into $\mathbb{C}^2$.
Define N as small tublar neighbourhood around $\mathbb{C} \times \{0\}$. Now $\mu(M)$ intersects δN at a knot which is isotopy equivalent to the knot K_n described in **Lemma 3.2**. Hence we have constructed a smooth Mobius strip bounding K_n. But this is in contradiction to *Lemma 3.2*.
Therefore our assumption must be incorrect. γ has an inscribed rectangle with an aspect ratio $= tan(\frac{\pi k}{2n})$

So from this fact, if we have $n = 3$ we get an aspect ratio of $tan(\frac{1}{6\pi}) = \frac{1}{\sqrt{3}}$ or $tan(\frac{2}{6\pi}) = \sqrt{3}$. Hence, every smooth Jordan curve has an inscribed rectangle with an aspect ratio $\sqrt{3}$. □

References

1. Cole Hugelmeyer *Every smooth jordan curve has an inscribed rectangle of aspect ratio* $\sqrt{3}$ `https://arxiv.org/pdf/1803.07417.pdf`
2. Richard Evan Schwartz *Inscribed Rectangle Coincidences* `https://www.math.brown.edu/reschwar/Papers/coincidence.pdf`
3. Josh Greene *The Rectangular Peg Problem* `https://www.birs.ca/workshops/2020/20w5088/files/BIRS`$_{G}$*reene.pdf*

Undergraduate,Mathematical Science NUIG

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

THE SYMBOLIC SIGNIFICANCE OF THE MÖBIUS STRIP

REMUS ARITON, JAMIE LARKIN, AND SURNAÍ Ó MAOILDHIA

(Communicated by Shelly Harvey)

ABSTRACT. "Years after Möbius' death, the popularity and applications of the strip grew, and it has become an integral part of mathematics, magic, science, art, engineering, literature and music. The Möbius strip is the ubiquitous symbol for recycling where it represents the process of transforming waste materials into useful resources. Today, the Möbius strip is everywhere, from molecules and metal sculptures to postage stamps, literature, technology patents, architectural structures, and models of our entire universe" [14].

Having stumbled upon this quote, our curiosity was sparked! How could the Möbius Strip have shaped art, music and literature? In what way had non-topologists interpreted its symbolism? Had it transcended topology completely and worked its way into the media unbeknownst to those that use it? We decided that the symbolic significance of the Möbius Strip had to be addressed.

In this essay, we use a range of different books, articles and online resources to explore our chosen topic. There are many articles showcasing singular examples of the Möbius Strip in the media and culture. Specifically, we explore how the Möbius Strip has influenced the shape, composition and narrative of literature and music - as well as its application in the engineering of products.

1. The Möbius Strip in Literature

The Möbius Strip, a "geometrical curiosity that has only one surface and one edge" has managed to find a prevalent place within literature [5]. Considering the circle has often been used in the shaping of plots – a simple example is that a story may come full circle -, it is not too surprising that the more unusual Möbius strip can provide a similar, and arguably more interesting format. Clifford A. Pickover writes that the Möbius strip "has become a metaphor for change, strangeness, looping, and rejuvenation" [13].

Martin Gardner explains the peculiarities of a "Double Moebius band" by describing a bug's passage: "A bug crawling between the bands could circle them indefinitely, always walking along one strip with the other strip sliding along its back. At no point would he find the "floor" meeting the "ceiling"" [5]. Ostensibly, it is an ordinary loop. However, as Gardner continues to explain, if there were to be a mark left by the bug's feet, when he completed a cycle he'd find the mark "not on the floor but on the ceiling, and it would require a second trip around the bands to find it on the floor again!" [5]. Generally, in regards to literature, the ant is the protagonist, trapped within a confounding cycle. The Möbius strip is a circle with a literal twist, so any use of the Möbius strip as a literary metaphor will have the cyclical nature of a circle, but with some fundamental difference within the

2020 *Mathematics Subject Classification.* Primary 57N25.

repetition: the ant returns to the same place, but upside down. 'In literature and mythology, the Möbius metaphor is used when a protagonist returns to a time or place with an alternative viewpoint, because a true Möbius strip has the intriguing property of reversing objects that travel within its surface.' [13].

For instance, the novelist, Gabriel García Márquez, subverted the typical ideas of time as simply linear, or circular, and instead used the more complicated Möbius strip as a model. Take the ending of the *One Hundred Years of Solitude* as an example: "[Aureliano] began to decipher the instant that he was living, deciphering it as he lived it, prophesying himself in the act of deciphering the last page of the parchments, as if he were looking into a speaking mirror" [10]. As though he were in two places at once; on a loop with only one side. In her article 'The Timeless Journey of the Möbius Strip', Serena Alagappan quotes the novel as well: "Úrsula sighed. 'Time passes.' 'That's how it goes,' Aureliano admitted, 'but not so much.'" [1]. Time is in the shape of a Möbius strip, and it provides a pathway within the novel, and within the lives' of the characters, on which these persisting ideas and themes can travel; the novelist illustrated the "human condition as a febrile dream in which love and suffering and redemption endlessly cycle back on themselves on a Möbius strip in time" [6]. These themes return again and again within the lives of the family, but in altered forms. In his novel, *Love in the Time of Cholera*, Marquez had love returning and returning within his characters' lives, but each time in a different form: "giddy adolescent love, mature love, romantic love, sexual love, spiritual love, even love so virulent it resembles cholera in its capacity to inflict pain" [6]. It is a recurring cycle, but with the half-twist that causes the ant to arrive on the roof, rather than the floor. Michiko Kakutani also writes that at the heart of this writer's work is "how the histories of continents and nations and families often loop back on themselves; how time past shapes time present".

The Möbius strip has represented many different things within literature. It may denote a "decidedly uncanny and troublesome [. . .] sort of infinity" [7]. Because the strip is a closed loop, there is a sense of entrapment within this infinity. The ant will continue to walk and, though the orientation will change, the ant will never be free of the strip itself. It "describes confinement within a three-dimensional perimeter: the experience of constant motion fused with a perpetual inability to cross to the other side" [7]. In Madame Bovary, the shape is apparent in the protagonist's "repeated gravitations to outsides that turn out to be extensions of her eternal inside" [7]. There are elements of contradiction and confusion to the Möbius strip as a metaphor, "with its two sides that are really one, or one side that to the untrained eye seems to be two, turning in a closed loop. . . " [7]. The strip has also been named "an apt allegory for losing control" [1]. In Alagappan's article, she writes: "The figurative and narrative implications of the Möbius strip are rich: when you try to go forward, you ring sideways, when you try to circle in, you find yourself outside" [1]. The mindboggling and bewildering Möbius strip is capable of signifying more than just one thing, making it an intriguing metaphor to use. Of course, it has also been said that the "most common contemporary use of the term "Möbius strip" occurs when alluding to any kind of mysterious looping behaviour", perhaps undermining, or underestimating, the complexities of the Möbius strip [13].

All the same, its singularity lends to it being quite an irresistible symbol; there's too much potential to ignore. "Many pairs of ideas and phenomena, conscious and unconscious, reality and fantasy, author and narrator, etc, are compared to the

Möbius strip. . . " [7]. The reason for this is that none of these pairs are disjoint. Instead, they are fused together: the narrator's thoughts coming from the thoughts of the author; conscious decisions stemming from the unconscious. Of course, these will turn back: whatever decision is made consciously will stir up that which is unconscious. Mathematical shapes have always been used for metaphor. Consider the circle of life. This ecological idea that life loops, starting and ending, only to start and end again elsewhere. However, with the movement from life to death, there is a crucial transformation. Life may indeed be a circle, as new life begins once old life ends. However, to include death within this figure, the Möbius strip - depending on one's belief -, could be a more fitting metaphor. "If you were to trace both "sides" of a Möbius strip, you would never have to lift your finger" [1]. We move from life to death, and perhaps back again, just as an ant walking along the Möbius strip. But this is just another example of the inexhaustible symbolism the Möbius strip has to offer; a symbolism that literature has made very compelling use of.

2. The Möbius Strip in Music

The Möbius Strip has had a profound effect on the history of music and has not only shaped the way we listen to music, but the way we compose it. We will look at how the Möbius Strip has influenced composers of the past and how it can be used as a creative tool for composers of the present and future.

In western music, we have 12 distinct notes (C, C#, D, D#, E, F, F#, G, G#, A, A# and B) [15]. As topologists, we believe that all C's are equal. Middle C, C3, C5, they are all C's. Similarly, all B's are equal, all D#'s are equal and so on. For this reason, we will ignore octaves, work modulo 12 and refer to each note as a different 'Pitch Class'. Musicians refer to the distance between two notes p, q (i.e. —q-p— mod 12) as an 'Interval' [15]. Playing two notes at the same time will give us a diatonic chord (two note). But how does this relate to topology?

To make things a bit more mathematical, let's plot our pitch classes on a 2-dimensional graph with (C, C) being the origin. As we move along the horizontal, we change the first note. As we move vertically, we change the second note. This graph has a few different properties. The main diagonal represents unison chords where the same note is being played twice. Any line y=x+c with constant c will represent some constant interval. —q-p— = 2 (major second), —q-p— = 5 (tritone), etc. [15]. Finally, any line perpendicular to the main diagonal will display contrary motion which is a musical progression of two voices moving in opposite directions [3]. This type of motion adds to one axis what it subtracts from the other [15].

Okay, we're getting more mathematical but let's try get specific to topology. Moving forward, it may be easier to visualise the graph if we rotate it 45° clockwise so that we have parallel motion on the main horizontal (i.e. first and second notes are the same) and perfect contrary motion on the main vertical. We also have motion of the first voice along the NE/SW diagonal and motion of the second voice along the NW/SE diagonal [16].

If we consider the four quartiles surrounding the origin. We run into issues as the same diatonic chord appears in multiple locations. For example, C-E and E-C occupy two different spots when ignoring octave and working modulo 12, they should be equivalent. This can be fixed by realising that the two lower quartiles are inversions of the upper two. We fold the plane in half along its main horizontal. Now notice that the left and right quartiles are also inverses of each other. If we cut each quartile in half along the vertical and take the resulting middle piece, we have a graph where every two-note combination is only represented once. In order to make this graph modulo 12, we may twist and connect the sides [16]. The resulting shape is a Möbius Strip!

This resulting shape is an incredibly useful tool for composers. It contains every possible path between diatonic chords. It also showcases some incredible properties. Recall that a Möbius Strip has only one edge. This edge represents the unison chords. The middle line of the strip represents the tritone which has an interval of 5. This splits an octave in half. As we move towards this line our intervals will increase (i.e. notes will become further apart). As we move away, they will decrease and become closer. Each line that runs parallel to the middle will carry a different tonality. Directly above, we have the perfect fourths. These will be followed by major thirds, minor thirds, major seconds and minor seconds before reaching unison at the edge. Chords on the same vertical line can be linked through perfect contrary motion and since the sum of a chords pitch classes is preserved, chords on the same vertical also sum to the same value modulo 12. For example, (C,F#) = 0 + 6 = 6, (G,B) = 7 − 1 = 6, (Ab,Bb) = -4 − 2 = -6 = 6. Finally, moving along the 45° line still alters only one note but we can no longer discern “first note” and “second note” as order no longer matters [17].

This is a tool that composers of the modern day can use. However, even before its discovery the Möbius Strip has been very present in music. We will take, for example, the works of Johann Sebastian Bach and how the Möbius Strip creeped into his scores more than a century before its official discovery.

Bach, like many composers, relied heavily on geometry to write his pieces. The composition process is very mathematical. Consider a musical score. We can view it like a 2-dimensional graph. The first dimension being time which is represented by the horizontal axis. The second being pitch which is displayed on the vertical axis. Bach would showcase this geometry in many of scores but none more so than his canons [2].

In musical terms, a canon is a piece of music in which a second voice imitates a lead voice after a delay [11]. This property means that the score for any canon can be mapped topologically onto a cylinder. If, for example, we look at Bach’s canon 3 from BWV 1087, we will notice that the lead voice sings the first 8 notes of the Goldberg Ground before the second voice joins singing the same melody but inverted. Going up in pitch when the lead goes down and vice versa. This score could clearly be mapped onto a cylinder as it will repeat until stopped at some arbitrary point [2].

Canon 5 from the same BWV 1087 has a much more interesting topology. Here, we have a total of four voices. Two high and two low. While the lower voices sing canon 3, the higher voices sing another canon in which the following voice joins in the third measure singing the same melody as the lead but inverted. Let’s look first at the high voices. When we invert the score, we see that the first two measures of the inversion are identical to the last two measures of the normal and visa versa [2].

This score is said to have ‘Glide-Reflection Symmetry’ which in 2-dimensional geometry, is a symmetry operation that consists of a reflection over a line and then translation along that line, combined into a single operation [8]. This is great news for us as any periodic text with glide-reflection symmetry can be mapped onto a Möbius Strip!

When we morph this score into a Möbius Strip, we find that it can be played in any direction. We can play both voices forward, both voices backwards, the lead voice forward and following voice backwards or the lead voice backwards and following voice forward. Regardless of how we choose to play this piece, the result will always be a functioning canon [2].

It is important to note that the score must be transparent. Which is to say that, given any point on the Möbius Strip, the score is identical on the front and back sides [2]. We can also create a Möbius Strip with the bass voices as they too showcase glide-reflection symmetry. However, a separate Möbius Strip is required as the axis of symmetry is different to that of the higher voices.

The influence that the Möbius Strip has had on the works of Bach is the truest testament to its cultural and symbolic impact on the world of music. A topological shape that has been present in compositions long before its discovery and continues to be implemented nearly two centuries after cannot be overlooked.

3. The Möbius Strip in the Engineering of Products

I am going to talk about the applications of the mobius strip in engineering of products because of the structural properties that the strip holds . First of which being 8-track tapes which . It has an endless tape with one twist, giving the basic tape-position configuration space the nature of a Mobius strip, getting you twice as much recording surface for the length. This was very useful because it meant that twice the amount of music can be played from the same length tape meaning the average cassette size could have twice longer music time. The 8-track tape is a magnetic-tape sound recording technology that was popular from the mid-1960s to the early 1980s. Bill Lear and his corporate cronies from Ampex, Ford, GM, Motorola, and RCA put their heads to together, they came up with a brilliant solution to the skipping experience of playing vinyl in the car. This would then change the world of music and cars for the future . "The suits rolled out their new eight-track tapes on September 15, 1965. Three of Ford's cars that year came with players as an option" [4]. There weren't even home consoles yet. By '67, all Fords had the option. The technology behind this was remarkable however there was room for improvement and this is what the famous cassette was based on , the 8-track cartridge format has its issues. Albums must be divided into the four discrete channels (segments), meaning songs are sometimes interrupted as the channel changes. The tape inside the cartridges can sometimes break, stretch and get tangled. The cassette came after 20 years because despite the structural issues and the mobius strip tapes were still used in some models before better, more convenient alternatives came about.

Printer and typewriter ribbons had a twist that allowed them to be cycled through a second time. The inking ribbon has a half twist forming a Mobius loop configuration which subsequently doubles its effective length . It wasn't so much the direction of the ink-flow through the ribbon, but the fact that the print head was offset, so on the second pass, it would use a different part of the ribbon. In the past, when the ribbons were unusable anymore it had to be replaced manually which was often dirty and time consuming. Because of this, a need was appreciated for a faster and more convenient way of changing ink ribbons. Therefore, endless mobius belt ink ribbons were mounted in cartridges which were then attached to a typing machine. This reduced the time required to change ribbons and furthermore insured a clean, easy quick process. “Since these ink ribbon cartridges had to be changed often when in heavy use, a means was provided for continuously depositing a layer of ink on the ribbon after it had moved past the hammer strike area of the typing machine” [1]. This allowed the ribbon to be used many times until the ribbon fabric wore out. However, because of the constant use of typewriters and other printing devices today, there is a need for an even longer lasting and more efficient ink ribbon. The use of the mobius strip in printer and typewriter ribbons revolutionised printing , “yielding synergistic ink capacity increases, long print life, long ribbon durability, and increased butt weld strength of the ends of the ribbon” [12].

In 1957, the B. F. Goodrich Company patented a conveyor belt that it went on to produce as the Turnover Conveyor Belt System. Incorporating a half-twist, it had the advantage over conventional belts of a longer life because it could expose all of its surface area to wear and tear. This meant that mobius belts had more than double life because usage was distributed evenly for the whole strip not just one of two sides of the belt. “Other patents of this kind that arise around the world include new designs of capacitors, surgical abdominal retractors, and self-cleaning filters in dry cleaning machines” [9]. Möbius strip belts are no longer manufactured because untwisted modern belts can be made more durable by constructing them from several layers of different materials.

The Möbius strip has also been tailored to various artistic and cultural products. Paintings have displayed Möbius shapes, as have earrings, necklaces and other pieces of jewellery. The green, three-arrowed universal sign for recycling also incorporates the Möbius band. There’s a depth in the meaning to the image that reminds you to reduce, reuse and recycle, as an endless loop. We can see that throughout history the incorporation of the mobius strip in products and patents had offered great usability and revolutionised some aspects of life making it a very interesting part of our history and how we explored the uses of this mysterious mathematical structure .

References

1. Serena Alagappan, *The timeless journey of the möbius strip.*
2. Eric L. Altschuler and Anthony Philips, *”the sound of topology: two-dimensional manifolds in bach ”.*, The Musical Times **156** (2015), no. 1933 (2015), 57–64.
3. Merriam-Webster Dictionary, *Contrary motion.*
4. History Garage, *The glorious rise and fall of the in-dash 8-track player.*
5. Martin Gardner, *Hexaflexagons, probability, paradoxes, and the tower of hanoi*, Cambridge University Press, 2008.
6. Michiko Kakutani, *Entwining tales of time, memory and love.*
7. Susanna Lee, *”madame bovary” and the möbius strip: Mapping a ”monde à part”.*, Nineteenth-Century French Studies **40** (2011), no. 1/2 (2011), 96–111.
8. George E. Martin, *Transformation geometry: An introduction to symmetry*, Springer Publishing, 1982.
9. Maxprog, *Engineering and application of mobius strip in practice.*
10. Gabriel García Márquez, *One hundred years of solitude*, Penguin Group, 1972.
11. The Editors of Encyclopaedia, *Canon.*
12. Justia Patents, *Mobius strip patents.*

13. Clifford Pickover, *The möbius strip: Dr. august möbius's marvelous band in mathematics, games, literature, art, technology, and cosmology*, Thunder's Mouth Press, 2006.
14. Clifford A. Pickover, *The math book*, Sterling Publishing Co., Inc., 2009.
15. Dmitri Tymoczko, *A geometry of music: Harmony and counterpoint in the extended common practice*, Oxford University Press, 2010, 2010.
16. ______, *A geometry of music: Harmony and counterpoint in the extended common practice*, Oxford University Press, 2010, 2010.
17. ______, *A geometry of music: Harmony and counterpoint in the extended common practice*, Oxford University Press, 2010, 2010.

4. Contributions

'The Möbius Strip in Literature' was written by Surnaí Ó Maoildhia; 'The Möbius Strip in Music' was written by Jamie Larkin; 'The Möbius Strip in the Engineering of Products' was written by Remus Ariton.

The project was typset by Surnaí Ó Maoildhia and Jamie Larkin.

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

MAPPER CLUSTERING ALGORITHM

MEGAN MCGLINCHEY, NICHOLA ROGERS, AND LIJUN ZOU

(Communicated by Julie Bergner)

ABSTRACT. In this article we explain the motivation, construction and application of the Mapper clustering algorithm to MA342 students. We first highlight the benefits of data visualisation in dealing with complex data sets. We then show the three advantages of using topology in data visualisation. We introduce some topological concepts which are not covered in MA342 syllabus such as nerve, Leray's theorem, refinement of a cover, Čech complex and Vietories-Rips complex, and explain how they motivated the construction of the algorithm. In particular, they can be used to deal with the practical issues such as the expensive computation costs of the algorithm. We describe the algorithm itself. The algorithm visualizes complex high dimensional data by reducing it to a simplicial complex that has similar geometric features. The algorithm is implemented with R-studio and an application of the algorithm is presented in which it provides meaningful information in medical prognosis.

1. Introduction

This project outlines an introduction to the Mapper clustering algorithm, designed by Singh, Memoli and Carlsson [SMC07]. The Mapper algorithm is a relatively new method of qualitative analysis, having only been developed in 2007. However, it has become widely utilized among mathematicians and data analysts alike. The intention of this algorithm is to transform high dimensional data sets to simplicial complexes, which have a much smaller number of points. By doing so, it enables visualizations to be completed, allowing any possible topological or geometric information to be identified.
The article is organized as follows. In section 2 we explain the motivation and importance of using topology in data analysis, which are key ideas to the Mapper. In section 3 we explain the Mapper clustering algorithm by first investigating its topological background and motivation, with an emphasis on the use of MA342 content in the construction of the algorithm, and then describing the algorithm itself. In section 4 we give an example using the TDA package available through R-Studio. In section 5 we give an application of the algorithm.

2. Topological Data Analysis (TDA) and Mapper

2.1. **Motivation.** In the process of data analysis there are a number of reasons why analysis can become difficult, such as volume of data, number of dimensions and also the complexity of the data. The motivation behind this algorithm being

Received by the editors May 15, 2021.
2010 *Mathematics Subject Classification.* Primary 55-04, 62-07; Secondary 55U05, 55U10.

created is clear; there are times when the original data can struggle to provide meaningful insights.

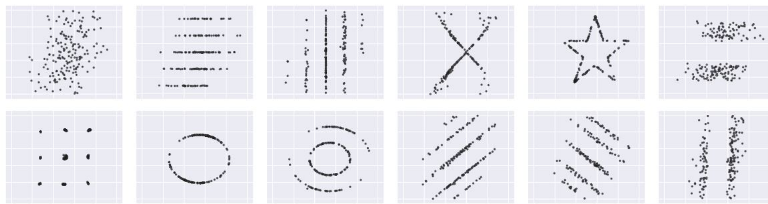


FIGURE 1. A collection of data sets with the same summary statistics, created by Matejka and Fitzmaurice [MF17].

It would be typical for mathematicians or analysts to complete summary statistics on a data set to gain a basic understanding of the data being used. This may include averages, standard deviations, correlation values, etc. However, often it is not until the analyst can visualize the data that the true meaning can become clear. This issue is highlighted in detail by Matejka and Fitzmaurice [MF17], where a vast range of shapes and distributions all have the same summary statistics as seen in Figure 1 [MF17]. The analysts would not see the true distribution of the data until it is visualized.

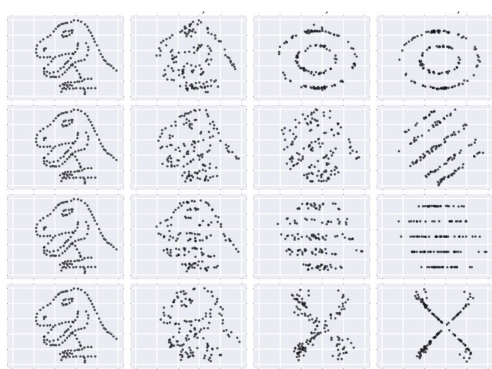


FIGURE 2. An additional data set with the same summary statistics, created by Matejka and Fitzmaurice [MF17].

A possibly more entertaining data visualization that has the same summary statistics as seen in Figure 2, to two decimal places is the "Datasaurus" [MF17], further highlighting the necessity of data visualisations.
Despite the obvious benefits for visualising data, there can occur times when a visualisation does not yield any insights. This is due to the complexity and size of the data set, making it much too difficult for the user to distinguish any groups or clusters of data, identify trends or arrive at any conclusions. To counteract this, the Mapper algorithm was created.

2.2. **Why topology?** As described in detail by Lum et al. [LSL$^+$13], there are three fundamental concepts of topology which are key to the use of the Mapper algorithm. The first concept is that topology is the study of shapes without coordinates; that is that they do not rely on a coordinate system, but rather on the distance which defines the shape. This allows for comparison of shapes from different coordinate planes or systems. The second idea is that the basic properties of shapes do not change from being skewed slightly. This could be best described by looking at the letter "B". By changing font type, or change of handwriting, it can still easily be recognised as the letter B, demonstrating how skewing the shape in small ways maintains the properties of the shape. The third key concept explained by [LSL$^+$13], is that shapes can be identified using a finite number of triangulations, which means a shape can be identified using a simplicial complex. This means that a shape of immense size can therefore be presented using a much smaller number of points, while maintaining the distinguishing features of the shape, such as loops.

3. Mapper Clustering Algorithm

3.1. **Topological background and motivation.**

Definition 3.1 ([Ale28])**.** Let A be an index set. Given a topological space X and a finite open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, the *nerve* of the cover $\mathcal{U}$ is the simplicial complex $N(\mathcal{U})$ whose vertex set is A with a k-simplex $\{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ whenever $\bigcap_{i=1}^{k} U_{\alpha_i} \neq \emptyset$.

Example 3.2. Let X be S^1 with usual topology. Suppose S^1 has an open cover $\mathcal{U} = \{U_1, U_2, U_3\}$, where each U_i is an arc of the circle and intersect with adjacent U_j [Wik21]. The resulting nerve $N(\mathcal{U})$ is shown in Figure 3. Note the 2-simplex $\{\{1, 2, 3\}\}$ is not in $N(\mathcal{U})$ since there is no common intersection among all three sets [Wik21].

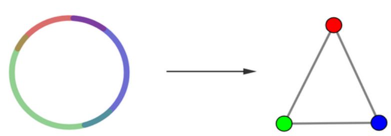


Figure 3. Left: S^1 covered by $\mathcal{U}$. Right: Nerve of $\mathcal{U}$.

Remark 3.3. We can define a map $f \colon X \to N(\mathcal{U})$. See [SMC07] for details.

Example 1.2 successfully transforms X into a simplicial complex that shares similar topological features. However, not every $N(\mathcal{U})$ reflects the topology of X accurately. To make sure we can extract geometric information of X from $N(\mathcal{U})$, we introduce the following theorem developed by Leray [Ler50].

Theorem 3.4 ([Ler50])**.** *Let $\mathcal{U}$ be an open cover of a compact space X such that every intersection of finitely many sets in $\mathcal{U}$ is either empty or homotopic to a set consisting of one point. Then X is homotopy equivalent to the nerve $N(\mathcal{U})$.*

The proof of the theorem is beyond the scope of MA342 content. Readers can see [SMC07] for details. Leray's theorem gives sufficient conditions for finding a simplicial complex that provides information of interest in some sense. However, there are three issues in its application.

The first problem is that given an open cover $\mathcal{U}$ of a topological space X, the nerve $N(\mathcal{U})$ may not be a 'good' enough approximation to X [Bra12]. To overcome it, we look at a 'smaller' open cover of X.

Definition 3.5 ([SSS78])**.** Let X be a topological space with two different coverings $\mathcal{U} = \{U_\alpha\}$ and $\mathcal{V} = \{V_\alpha\}$. Then $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if every element of $\mathcal{V}$ is a subset of some element of $\mathcal{U}$.

Remark 3.6. We say $\mathcal{V}$ is *finer* than $\mathcal{U}$ and $\mathcal{U}$ is *coarser* than $\mathcal{V}$ [Pro21].

Proposition 3.7 ([Bra12])**.** *If an open cover $\mathcal{V}$ is a refinement of $\mathcal{U}$, there exists a unique map between their underlying topological spaces up to homotopy.*

See [Bra12] for proof. This proposition suggests that the topological space induced by $N(\mathcal{U})$ is homotopic equivalent to some underlying subspace of $N(\mathcal{V})$ [Bra12]. Thus, the refinement captures all the topological information of the original cover and gives a better approximation of X. In practice, we choose an open cover and often investigate its refinements with different coarseness levels, and the features that exist among most of the values are more likely to exist in X [SMC07].

The second problem is that we are often unable to find a open cover $\mathcal{U}$ as we do not know the shape of X. We can use what we learned in MA342 to solve this problem. The idea is developed by [SMC07]: Define a map $f\colon X \to Z$ where Z is a known topological space with an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$. Since f is continuous, the pre-image $f^{-1}(V_\alpha)$ is open in X and thus $\mathcal{U} = \{f^{-1}(V_\alpha)\}_{\alpha\in A}$ is an open cover of X [SMC07]. Note we can form a refinement for $\mathcal{U}$ by writing each $f^{-1}(V_\alpha)$ as a union of its disjoint connected components. We say these connected components form a *refined pullback* for X.

The third problem is that, computation of nerves is complicated, and with high costs. To check for the existence of a 6-simplex, we need to check common intersections of any six open sets [Car09]. To overcome this issue, we introduce the following definitions, which are taken from [CM17].

Definition 3.8 ([CM17])**.** Let X be a metric space, and $\epsilon > 0$. The *Čech complex* $Cech_\epsilon(X)$ is the set of simplices formed by the elements of X such that a k-simplex $\{x_0, x_1, ..., x_k\}$ exists if the intersection of $k+1$ balls $B(x_i, \epsilon) = \{x \in X : d_x(x, x_i) < \epsilon\}$ is non-empty.

Definition 3.9 ([CM17])**.** Let X defined as above. The *Vietories-Rips complex* $Rips_\epsilon(X)$ is the set of simplices such that a k-simplex $\{x_0, x_1, ..., x_k\}$ exists if $d(x_i, x_j \leq \epsilon)$ for all (i, j).

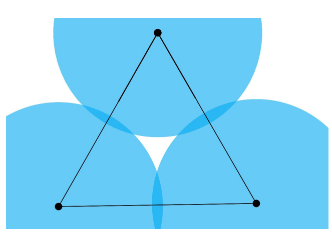


Figure 4. $Cech_\epsilon(X)$.

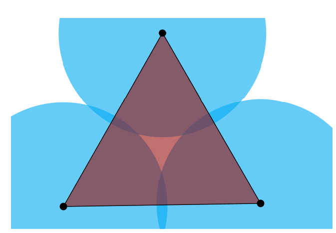


Figure 5. $Rips_\epsilon(X)$.

Remark 3.10. $Cech_\epsilon(X)$ is the nerve of a set of metric balls [Ghr14]. It is not hard to see that $Cech_\epsilon(X)$ satisfies Leray's theorem since we proved in MA342 that any convex set is homotopic equivalent to a singleton space $\{1\}$.

Remark 3.11. $Rips_\epsilon(X)$ may not satisfy Leray's Theorem.

For a $Rips_\epsilon(X)$ we add a k-simplex by calculating the distance between any pair of $k+1$ vertices, while for $Cech_\epsilon(X)$ we need to compute the intersection of $k+1$ balls [Car09]. Figure 4 and Figure 5 illustrate the idea. Intuitively, computation cost of $Rips_\epsilon(X)$ is lower than that of $Cech_\epsilon(X)$. This fact motivates us to use $Rips_\epsilon(X)$ to approximate X and the following proposition is introduced.

Proposition 3.12 ([CM17])**.** *For any* $\epsilon > 0$, $Cech_\epsilon(X) \subseteq Rips_\epsilon(X) \subseteq Cech_{2\epsilon}(X)$.

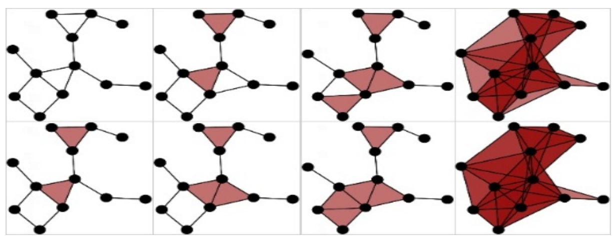


FIGURE 6. Left to Right: Values of ϵ ranging from 0.1 to 0.2.
Top: The Čech complex. Bottom: The Vietoris-Rips complex.

Instead of giving a proof, we illustrate the theorem with Figure 6. By observation the dimension of $Rips_\epsilon(X)$ grows quicker than that of $Cech_\epsilon(X)$ does. Also, $Rips_{0.1}(X)$ (the left bottom) is a clique of $Cech_{0.2}(X)$ (the right top). The theorem suggests that, for a given metric space X and $\epsilon > 0$, if $Cech_\epsilon(X)$ and $Cech_{2\epsilon}(X)$ both share a piece of information of X, so does $Rips_\epsilon(X)$.
The ideas in the Section 3.1 combined to form the Mapper clustering algorithm.

3.2. **Mapper clustering algorithm.** The key idea of the Mapper clustering algorithm is to summarize given data set X through the nerve of a refined pullback of X under a suitable function $f\colon X \to Z$ [CM17, SMC07]. [Car09, CM17, SMC07] describes the algorithm in the following steps.

Step 1. Let X be the data set we want to investigate and Z be a known space. Choose a continuous function $f\colon X \to Z$ that captures the important information of X.

Step 2. Form an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ for Z. Note a small change of cover for Z can lead to a significant change in the image generated by Mapper. To gain stability of the method, we in general compare outputs corresponding to a range of covers with different coarseness level.

Step 3. Ensure $\mathcal{U} = f^{-1}(\mathcal{V})$ satisfies Leray's Theorem.

Step 4. Implement a standard clustering algorithm for each $f^{-1}(\mathcal{V}_\alpha)$. By doing so, a refined pullback is found as desired. Note any clustering algorithm works here and there is no assumption on the number of clusters in the beginning.

Step 5. Calculate and analyze the nerve of the refined pullback of X.

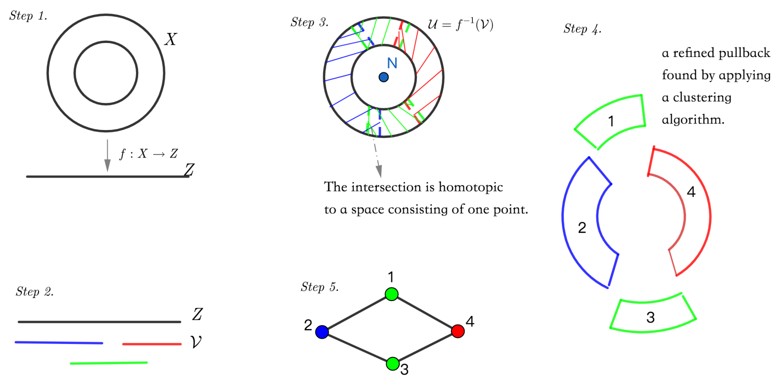


FIGURE 7. General flow of the Mapper clustering algorithm.

Figure 7 illustrates the general flow of the algorithm. The method considerably simplifies the original data set, but meanwhile its result provides certain information of the area of interest [SMC07]. In the next section, we will use R to see how this method works in reality.

4. TDA-mapper in R-Studio

The Mapper algorithm described above in detail is now so widely accepted as a data analysis tool that its functionality is available as a package for installation in R-Studio. Within the TDA-Mapper package, a dataset is available to allow students and data analysts alike to practice using the package and become familiar with the steps involved. This data included in the TDA-Mapper package is that of the Miller-Reaven diabetics study [SMC07]. The sample data included refers to patients identified as being "Chemical Diabetic", "Overt Diabetic", or "Normal" which implies they are not a diabetic patient. The sample data provided is of dimension 145x6, with five fields containing numerical data and the sixth field identifying diabetic status. The remaining fields within the data set are abbreviations of medical terminology and expert medical knowledge would be required to fully understand these fields, however, small descriptions of these fields are available in [SMC07].

As part of the algorithm implementation in R-Studio described by [CM16], it is possible to present the graph produced following the clustering procedure, as can be seen in Figure 8. However, the true value of TDA-Mapper in R-Studio does not become apparent until a interactive visualisation is created, which uses D3 as part of the javascript library [CM16]. Through the use of D3, the above plot of the data in Figure 9 transforms to have nodes of representative size and colour, while having the additional benefit of being interactive for the user through R-Studio. The interactions include zoom, maneuvering the nodes and edges, highlighting certain nodes, and hovering over nodes can provide further details.

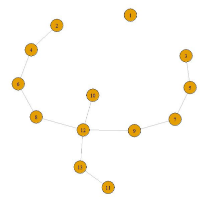


Figure 8. The data following the clustering algorithm.

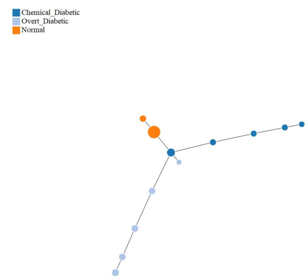


Figure 9. A still of the interactive visualisation produced through TDA-Mapper in R-Studio.

As can be seen in Figure 9, the visualisation produced does not match with that produced for [SMC07], however, it can still provide some interesting insights to the reader. The size of the node represents the size of the cluster, with Normal being the largest cluster. Interestingly, there is no edge present between Normal and Overt Diabetics. Although this is difficult to interpret for the author of this paper, this could prove interesting for a medical professional.

5. Applications

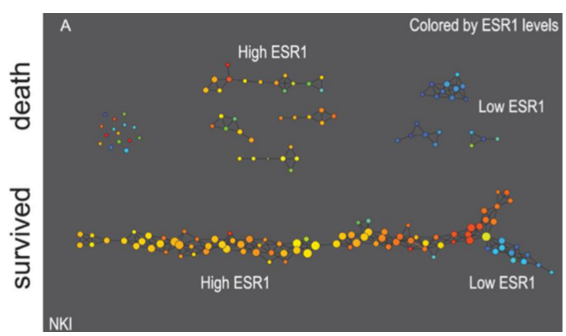


Figure 10. This network is constructed using gene expression and survival information, created by Lum et al. [LSL+13].

One of the more interesting applications of the Mapper clustering algorithm is its advancement in the identification of sub-populations in patients with breast cancer [BVD+20, LSL+13]. Topological maps can more finely stratify patients compared to classical methods and hence helps to identify the sub-groups for which targeted therapy is better suited. A high expression level of the estrogen receptor gene (ESR1) is positively correlated with improved prognosis, given that these individuals are likely to respond well to standard therapies [LSL+13]. There is also strong correlation between individuals with low ESR1 levels and a decreased survival rate.

Despite these findings, it is evident from Figure 10 [LSL+13] that there are some outliers in our network. The patients that survived form the horizontal 'Y' structure and those who did not survive form the smaller networks [LSL+13]. Some sub-groups with high ESR1 levels do not respond well to standard therapies. In addition, there are also sub-groups of patients with low ESR1 levels that have a high survival rate [LSL+13]. This is depicted as the blue branch of the horizontal 'Y' structure.

These sub-groups are much more easily identified using our method than classical approaches as they tend to separate points in the data set which are in fact close. The identification of these sub-groups using the Mapper clustering algorithm can then influence treatment choice and lead to an overall improved prognosis.

References

[Ale28] Paul Alexandroff, *Über den allgemeinen dimensionsbegriff und seine beziehungen zur elementaren geometrischen anschauung*, Mathematische Annalen **98** (1928), no. 1, 617–635.

[Bra12] Jeremy Brazas, *The cech expansion: nerves of open covers*, `https://wildtopology.wordpress.com/2012/10/30/the-cech-expansion-i-nerves-of-open-covers/`, 2012, [Online; accessed 21-May-2021].

[BVD+20] Quang-Thinh Bui, Bay Vo, Hoang-Anh Nguyen Do, Nguyen Quoc Viet Hung, and Vaclav Snasel, *F-mapper: a fuzzy mapper clustering algorithm*, Knowledge-Based Systems **189** (2020), 105107.

[Car09] Gunnar Carlsson, *Topology and data*, Bulletin of the American Mathematical Society **46** (2009), no. 2, 255–308.

[CM16] Frédéric Chazal and Bertrand Michel, *Mapper algorithm with the r-package tdamapper*, `http://bertrand.michel.perso.math.cnrs.fr/Enseignements/TDA/Mapper.html`, 2016, [Online; accessed 21-May-2021].

[CM17] ———, *An introduction to topological data analysis: fundamental and practical aspects for data scientists*, arXiv preprint arXiv:1710.04019 (2017).

[Ghr14] Robert W Ghrist, *Elementary applied topology*, vol. 1, Createspace Seattle, 2014.

[Ler50] Jean Leray, *L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue*, J. Math. Pures Appl **29** (1950), no. 9, 1–139.

[LSL+13] Pek Y Lum, Gurjeet Singh, Alan Lehman, Tigran Ishkanov, Mikael Vejdemo-Johansson, Muthu Alagappan, John Carlsson, and Gunnar Carlsson, *Extracting insights from the shape of complex data using topology*, Scientific reports **3** (2013), no. 1, 1–8.

[MF17] Justin Matejka and George Fitzmaurice, *Same stats, different graphs: generating datasets with varied appearance and identical statistics through simulated annealing*, Proceedings of the 2017 CHI conference on human factors in computing systems, 2017, pp. 1290–1294.

[Pro21] ProofWiki contributors, *Definition:refinement of cover*, `https://proofwiki.org/wiki/Definition:Refinement_of_Cover`, 2021, [Online; accessed 21-May-2021].

[SMC07] Gurjeet Singh, Facundo Mémoli, and Gunnar E Carlsson, *Topological methods for the analysis of high dimensional data sets and 3d object recognition.*, SPBG **91** (2007), 100.

[SSS78] Lynn Arthur Steen, J Arthur Seebach, and Lynn A Steen, *Counterexamples in topology*, vol. 18, Springer, 1978.

[Wik21] Wikipedia contributors, *Nerve of a covering — Wikipedia, the free encyclopedia*, `https://en.wikipedia.org/w/index.php?title=Nerve_of_a_covering&oldid=1005090868`, 2021, [Online; accessed 21-May-2021].

National University of Ireland, Galway, Ireland

National University of Ireland, Galway, Ireland

National University of Ireland, Galway, Ireland

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0894-0347(XX)0000-0

A BRIEF OUTLINE ON TOPOLOGICAL DATA ANALYSIS AND THE MAPPER ALGORITHM

PATRICK WING MCHALE, NICOLAS AMAYA AGUILAR, AND JOSHUA STONEY

Abstract. In this paper we will give an overview on the mapper algorithm, a tool commonly used in Topological Data Analysis (TDA). We will summarise and explain its process, highlighting the topology behind it, giving an overview into the algorithm. Furthermore, we will showcase its commercial applications from healthcare to finance.

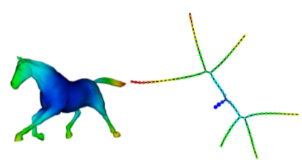


Figure 1. TDA on a 3D image of a horse

1. Overview of TDA

The 'Data Explosion' is an all too familiar problem faced in the worlds of science, engineering, and finance. The rate at which data is gathered and stored is ever growing and shows no sign of slowing down. 'Big Data' sets have taken over the world. This data often encrypts valuable data that can be easily lost in its size, complexity, or to noise. The challenge begins in extracting the useful information from these sets. Topological Data Analysis (TDA) has emerged as a helpful tool in the extraction of this information. So first the question that must be asked, what is TDA? TDA takes its birth from persistent homology. In 1990 Patrizio Frosini introduced a size function, which equates to the 0th persistent homology. It was nearly 10 years later when Vanessa Robins studied the images of homomorphism induced by inclusion. It did not take long for Edelbrunner to introduce the concept of persistent homology together with an efficient algorithm and its visualization as a persistent diagram.[1] Gunnar Carlsson, famed mathematician in topology, reformulated the initial definition and gave an equivalence visualization method called persistent barcodes. How does TDA help us with the problem of 'Big Data'? The simplest way to partially answer this is that shapes matter. Topology uses tools to compute geometrical features such as holes and components of a geometric objects. These tools are what we will use to get the relevant information from the

2010 *Mathematics Subject Classification.* Primary 62-07, 14F45 .
Edited by Julie Bergner.

(A) (B)

Figure 2

(A) (B)

Figure 3

data sets. However, as things stand, we can do very little with the data points in this set. A set X with n points has n component and 0 holes. For TDA to work we need these points to connect. To do this, we need to 'thicken' set X.

Imagine X as a finite set of points on a 2-D plane. Let d be a positive number and let T(X,d) be the set of all points in the plane a distance d from some point in X. We can think of this as a 'Thickening' of X.

Example: in Figure 2 (A) we can see there is a set of unconnected data points X. Figure 2 (B) shows T(X,d) in red for some chosen d. This forms three separate components of the data set. In the second example (Figure 3 (A) and Figure 3 (B)) we can see the thickening forms a ring.[2] This allows us to get an insight into the geometrical features of X by studying the topological properties of T(X,d). We chose a distance d such that when applied to a point in X a connection (edge) is made between those points. The choice of distance d is important. If d is too small, there may be multiple connecting components and small holes, also if it is too large all points get connected and we get a giant simplex.[3] To find out what is relevant and what is noise, we consider all values of d. As d grows holes appear and disappear. These are recorded on a barcode.

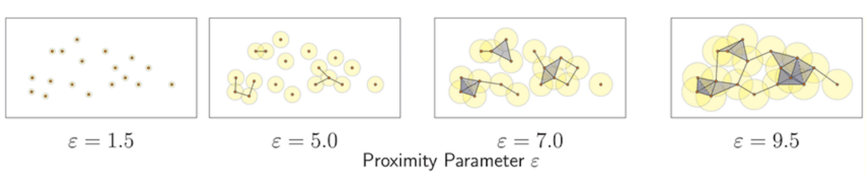


Figure 4

2. The Mapper Algorithm

Topological data analysis is a modern approach to analysing data, notably it examines for purpose behind the "shape" of data. The Mapper Algorithm is a technique used in topological data analysis. Simply, it examines a piece of high-dimensional data and forms it into insightful shapes with the intent to gain a higher understanding of the data. The algorithm uses a concept in topology known as homology, more specifically persistent homology to measure connected components and perform analytical actions on point clouds and simplicial complexes.

2.1. **How it works.** There are several steps to the mapper algorithm, converting the high dimensional data into a single dimension using a filter function, constructing a cover $(U_i)_i \epsilon I$ to form a set of overlapping bins, clustering the bins, representing each cluster with a vertex, creating the graph by forming edges between intersecting clusters. An example of the process is as follows:

2.2. **Mapping to lower dimensions.** In this example we create a graph representing a three-dimensional hand. In figure 5 we see point cloud data representing a hand. Essentially, a spread of points on the surface of the hand. The next step is to colour the data set through use of a filter function. The function we are using projects the three dimensions onto the x-axis. The colour from the function is translated onto the preimage.

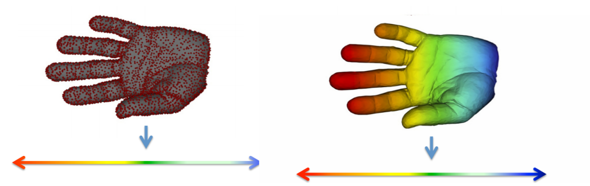


Figure 5. Colouring by filter value

2.3. **Forming overlapping bins.** A cover of overlapping, open intervals is then created over the image. Then overlapping bins are created on the preimage of the overlapping intervals. Data in the preimage is placed into bins corresponding to the intersecting open intervals on the image. Notice that there is data in the intersection of these bins, i.e., they have points in common. The bins are then separated from each other. See Figure 6.[4]

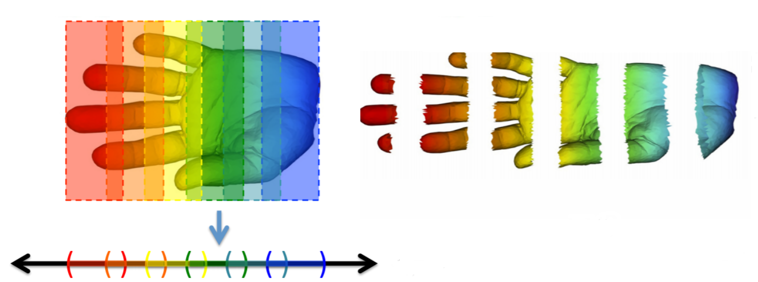


Figure 6. Overlapping Bins

2.4. **Creating the graph.** At this stage the bins can be clustered, using clustered components. We can see, in Figure 6, in the first bin we have 3 clusters (the fingertips), then in the following bin we have 4 clusters, and so on. Every cluster can be represented by a single vertex. Thanks to this, we can create a graph by joining vertices using edges to connect them when there exists a non-empty intersection between clusters. Finally, we are left with a graph representing our original data set.[5]

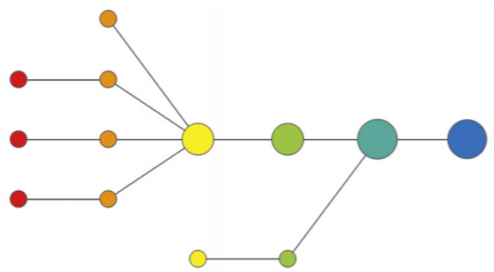


FIGURE 7. Graph of The 3D Hand

3. Strengths and Weaknesses of The Mapper Algorithm

Unfortunately, because of dimension reduction, the mapper algorithm suffers from projection loss. Points that would be considerably apart in higher dimensions may be projected close together. The algorithm compensates for this by acting upon the data in its high dimensional state after the filter function has been applied to it. This ensures that substructures in the original space are accounted for.[6]

About dimensional reduction, depending on the "shape" of the high dimensional data, often certain aspects of the data will be lost in the final graph depending on the plane in which the data is mapped to. As another example, we see the in Figure 8 that the mapper algorithm captures the horizontal flares of the blob but not the vertical flares. We lose this insight when we project to the x-axis.

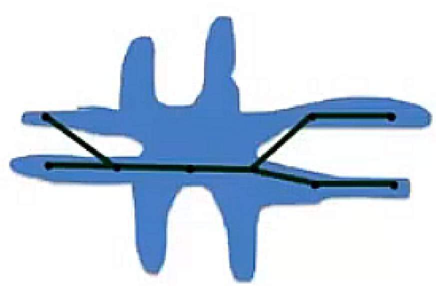


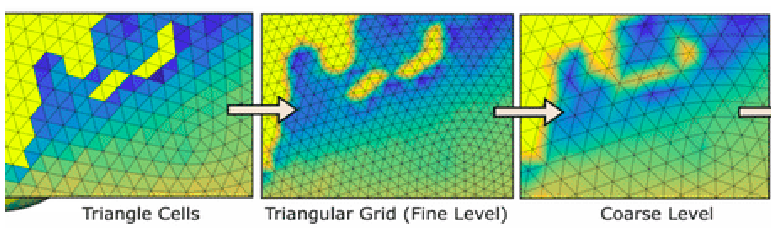
FIGURE 8. Mapper Algorithm Applied to a Shape

Of the many strengths the mapper algorithm brings to data analysis, one of the key features is its ability to showcase information in a compressed form. This makes the data more accessible to analysts. In addition, the mapper algorithm may make connections between seemingly unrelated areas of the data creating potential for further investigation. The mapper algorithm has high potential for furthering the field of data analysis through showcasing new possibilities.

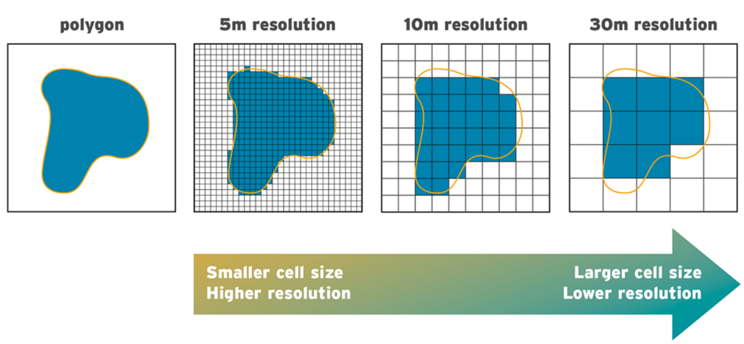
4. Applications of Topological Data Analysis

Topological Data Analysis (TDA) is an approach which aims to uncover hidden structure in data sets using techniques from topology, and other areas of pure maths. The driving motivation of TDA is to study and extract information from the shape of data. The main tool used in TDA is persistent homology. Persistent homology computes the topological features of a space by comparing the space at various spatial resolutions (i.e. how clear or blurry the data set is). The more common or 'persistent' features stand out over a wide range of resolutions. Persistent features which appear across a wide variety of resolutions are assumed to more likely represent 'true' features of the space. This method is used to extract the true topological features without prior domain knowledge.[7]

In figure 9 (A) we see how a change in resolution can distort 1 land mass and 1 island into 1 land mass and 2 islands (Triangular Grid), or into 1 land mass with no islands (Coarse Level). There are many applications of TDA such as Shape Study, Image Analysis, Viral Evolution, Progression Analysis of Disease, Visualisation and much more. TDA is still not a well-known topic for data mining. It has many applications in biology, economics, medicine, politics, physics, etc. Certain topics of interest include, but are not limited to clustering, pattern recognition, skeletonization (Figure 6), Mapper (Figure 7) and persistent homology.



(A) The Effect of Resolution on Connected Components



(B) The Effect of Different Resolutions on a Shape

Figure 9

4.1. **Applications of TDA to Breast Cancer.** In their paper "Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival" M. Nicolau, A.J. Levine and G. Carlsson introduce a method to extract information from high-dimensional and very sparse data. By using topology this method provides deeper insights than current analytic techniques. The method is an application of Mapper to transcriptional genomic data, data of genes, which make new copies of themselves (an error during this reproduction/transcription of genes/DNA is how cancers form). Their method was termed Progression Analysis of Disease (PAD). First it analyses data through cluster analysis, then goes deeper to find meaningful shape characteristics in the data. PAD also incorporates aspects of visualisation, which means it provides pictures & graphs which can be used to further explore the data.

The PAD method identified a unique subgroup of Oestrogen Receptor-positive (ER+) breast cancers, whose tumour cells have more Oestrogen receptors than healthy cells and so they grow faster with Oestrogen. The patients in the subgroup have 100% survival and no metastasis, i.e., the cancer does not spread. This subgroup (denoted c-MYB+) is a type of ER+ cancer, it has a clear and distinct, statistically significant signature, but is invisible to cluster methods and does not fit into the Normal-Like subtypes of ER+ breast cancers.

The values of the Mapper filter functions were then computed, the figure below shows the resulting graph for a particular set of parameters. The tumour cell clusters have been coloured, the colour describes how much the genes of the diseased tissue deviate from the genes of the healthy tissue. Blue clusters represent gene expression close to normal, red clusters represent a large deviation from normal gene expression (the red clusters are the ER+ branch to the right of the graph). The graph is made up of three branches representing different types of breast cancers: The ER- (negative) sequence, the ER+ (positive) sequence and the Normal-Like tissue, which is a subtype of ER+ breast cancer. Although the Normal-Like breast cancer subgroup was already known, the new c-MYB+ subgroup could now be identified by using Mapper to get the graph below.[8]

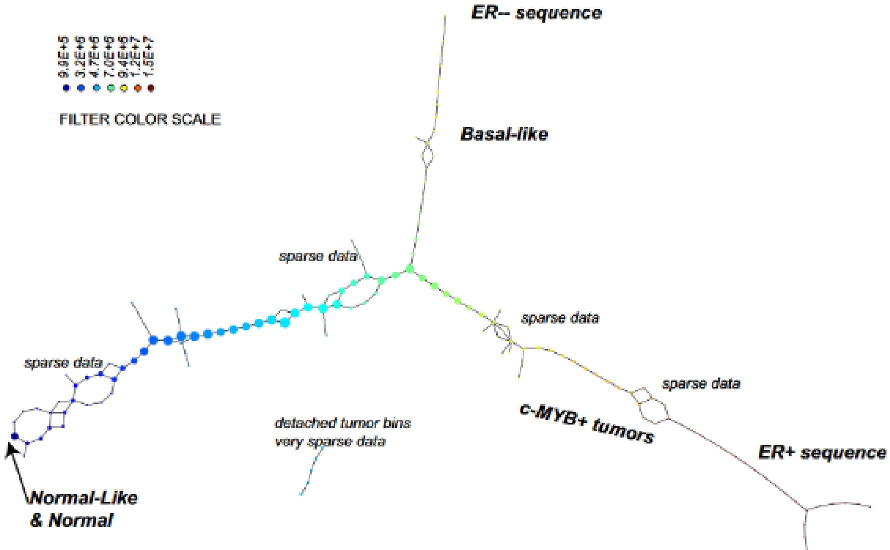


FIGURE 10. Mapper Algorithm Identifies Previously Invisible Cancer Subgroup

4.2. **Ayasdi.** Ayasdi is a machine learning and artificial intelligence software company that offers a software platform and applications to business and other organisations looking to analyse and build predictive models using big data or highly dimensional data sets. Governments and other organisations have used Ayasdi's software in a variety of cases including anti-money laundering, disease research, fraud detection, oil and gas well development, and trading strategies to name a few. Ayasdi uses supervised and unsupervised machine learning algorithms. Unsupervised learning helps with discovering the hidden structure in data. Supervised learning helps with the construction of predictive models.[9]

Ayasdi focuses on hypothesis-free, automated analytics at scale. The Ayasdi system takes in each data set, runs various supervised and unsupervised machine learning algorithms on the data, finds and ranks best fits and then applies TDA to find similar groups in the data. This is like persistent homology, but instead of one algorithm over various spatial resolutions of the data set, it runs various algorithms over the same data set. When compared with manual approaches to statistical analysis and machine learning, results with Ayasdi will typically be much faster to achieve and more accurate, reducing biases in data analysis. Rather than relying on analysts to manually run algorithms in support of pre-existing hypotheses, this method presents "what the data says" in an unbiased manner.[10]

4.2.1. *Ayasdi's Products.* Ayasdi has 3 main services they promote.

1.)Solutions to Financial Crime – money laundering, fraud, corruption, and trafficking.

2.)ALM: Risk, liquidity, and profitability – analysing customer behaviour to maximise liquidity.

3.)Real KYX Intelligence – analysing customer behaviour to predict risk, product interest, loyalty, etc.

Topology and TDA are well suited for analysing complex data with potentially millions of attributes. Ayasdi uses advanced analytics software to create highly interactive visual networks that allows them to explore and understand critical patterns and relationships in their data.[11]

References

1. Wright, M. (2015) Introduction To Persistent Homology
2. Lesnick, M. (2013) Studying the Shape of Data Using Topology
3. Singer, Z. (2019) Topological Data Analysis — Unpacking the Buzzword
4. Darcy, I.K. (2013) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology.
5. Lum, P. et al. (2013) Extracting insights from the shape of complex data using topology.
6. Minervino, M. (2020) Topological data analysis with Mapper
7. https://en.wikipedia.org/wiki/Topological_data_analysis
8. Nicolau, M., Levine, A.J., Carlsson, G. (2011) Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival
9. https://en.wikipedia.org/wiki/Ayasdi
10. Ayasdi (2015) TDA and Machine Learning: Better Together
11. https://www.ayasdi.com/

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

TOPOLOGICAL CONJUGACY

ALAN HENSON, BRIAN MCGUINNESS, AND THOMAS CONROY

(Communicated by Katrin Gelfert)

Abstract. Conjugacy is a familiar concept in all areas of mathematics, including topology. In this project we investigate the notion of topological conjugacy between maps on two topological spaces. We will explore the concepts of topological semi-conjugacy and local conjugacy. We will then apply topological conjugacy to various dynamical systems.

1. Background

1.1. **Definitions.** Firstly, we must outline some basic definitions which we will refer to in this text:

Definition 1.1. A function $f : X \to Y$ is said to be *injective* or *1 to 1* if $\forall\, a, b \in X$,

- f maps distinct elements $a, b \in X$ to distinct elements $f(a), f(b) \in Y$.
- $f(a) = f(b) \Leftrightarrow a = b$.

Definition 1.2. A function $f : X \to Y$ is said to be *surjective* or *onto* if $\forall\, y \in Y$,

- $\exists\, x \in X : f(x) = y$.
- i.e. for all elements $\in Y$, the space X contains its pre-image under f.

Definition 1.3. A function $f : X \to Y$ is said to be *bijective* if it is both injective and surjective.

Definition 1.4. A function $f : X \to Y$, where X,Y are topological spaces is said to be continuous if and only if the pre-image of every open set $\in Y$ is open $\in X$. Intuitively, we think of a continuous function as one where a small change in the input yields only a small change in the output.

Definition 1.5. A continuous bijective function $f : X \to Y$ between topological spaces is called a homeomorphism if there exists a continuous function $g : Y \to X$ such that

$$g(f(x)) = x \;\forall\; x \in X$$

and

$$f(g(y)) = y \;\forall\; y \in Y.$$

Definition 1.6. Two topological spaces X and Y are homeomorphic if there is a homeomorphism $f : X \to Y$.

Received by the editors 05/04/2021.
2020 *Mathematics Subject Classification.* Primary 37C15 .

2. Topological conjugacy

2.1. **Introduction.**

Definition 2.1. Two functions $f : X \to X$ and $g : Y \to Y$ which map topological spaces X and Y to itself are said to be topologically semi-conjugate if there exists a surjective function $h : X \to Y$ such that $h \circ f = g \circ h$. The functions f, g are defined as

$$x \mapsto f(x),\ y \mapsto g(y)$$

and

$$y = h(x),$$

such that

$$h(f(x)) = g(h(x))$$

for all $x \in X$. [1]

The idea of topological semi-conjugacy is linked very closely to that of topological conjugacy, with one distinct feature:

Definition 2.2. Two functions $f : X \to X$ and $g : Y \to Y$ are said to be topologically conjugate if they are both semi-conjugate and h is a homeomorphism, i.e. a continuous bijective function. Such a homeomorphism h is also called a topological conjugacy. [1]

Topological conjugacy is an equivalence relation and the set of all maps $X \to Y$ is divided into classes of topologically conjugate maps, thus f and g are related if they are topologically conjugate. We can rewrite

$$h \circ f = g \circ h$$

to read

$$g = hfh^{-1}.$$

This means that the function g applied to an element $y \in Y$ is equivalent to applying h^{-1} followed by f, and then applying h. See the diagram below to confirm this equivalence.

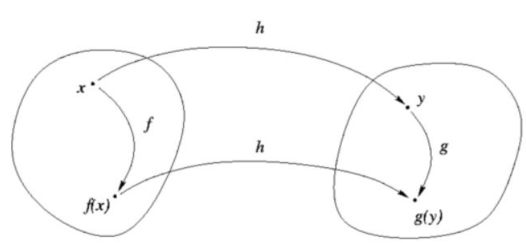


Figure 1. Topologically conjugate maps f, g and homeomorphism h between topological spaces X, Y. [4]

2.2. **Local Conjugacy.** It is also possible to localise the definition of topological conjugacy for open sets in topological spaces.

Definition 2.3. In a topological space X, an open neighborhood U of a point x is an open set containing x. A set containing an open neighborhood is simply called a neighborhood.

Definition 2.4. Two maps $f : X \to X,\ x \mapsto f(x)$ and $g : Y \to Y,\ y \mapsto g(y)$ are said to be locally conjugate near respective points x_0 and y_0 if there exists a homeomorphism $h : U \to V$ of an open neighbourhood U of $x_0 \in X$ onto an open neighbourhood V of $y_0 \in Y$, which satisifes $h(f(x)) = g(h(x))$ for all $x \in U$ such that $f(x) \in U$, and $y_0 = h(x_0)$. [4]

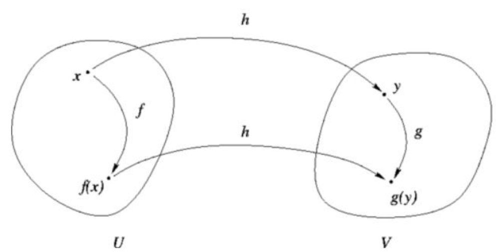


FIGURE 2. Topologically conjugate maps f, g and homeomorphism h between open neighbourhoods $U \in X$ and $V \in Y$. [4]

2.3. **Equivalence relation.**

Lemma 2.5. *Topological conjugacy is an equivalence relation.*

Proof. We can establish that topological conjugacy is an equivalence relation by proving reflexivity, symmetry and transitivity.

- *Reflexive*: We must show that there exists a h, where h is a homeomorphism such that $h \circ f = f \circ h$. Choose $h(x) = x$. Then $h(f(x)) = f(x)$. So $h \circ f = f$.
  Since $h(x) = x$, $f(h(x)) = f(x)$. Therefore, $f \circ h = f$. We can write f in the form $f = h^{-1} f h$. Then $f \sim f$, or f is topologically conjugate to itself, so topological conjugacy is reflexive.

- *Symmetry*: We must show that $f \sim g \Leftrightarrow g \sim f$. Suppose $f \sim g$. Then we can write $h \circ f = g \circ h$, or $g = hfh^{-1}$. Since h is a homeomorphism, we can rearrange this identity by composing both sides with h^{-1} to the left, which gives $h^{-1} \circ g = f \circ h^{-1}$ as required. Similarly, we can write this as $f = h^{-1} g h$. Then $g \sim f$, so topological conjugacy is symmetric.

- *Transitivity*: We must show that $f \sim g$ and $g \sim k \Rightarrow f \sim k$. Suppose $f \sim g$ and $g \sim k \Rightarrow f \sim k$. Then $g = hfh^{-1}$ and $k = pgp^{-1}$, where h, p are homeomorphisms. Substituting our identity for g into our identity for k we get $k = p(hfh^{-1})p^{-1}$. By the property of inverse composition of functions we know that $h^{-1} \circ p^{-1} = (ph)^{-1}$. From this we get $k = (ph)f(ph)^{-1}$. Then $f \sim k$, so topological conjugacy is transitive.

□

2.4. **Topological conjugacy in dynamical systems.**

2.4.1. *Iterated functions and orbits.*

Topological conjugacy is important in the study of iterated functions and more generally dynamical systems, since, if the dynamics of one iterated function can be solved, then those for any topologically conjugate function follow trivially.

The foundation of topological conjugacy being an equivalence relation is very useful in the theory of dynamical systems. Each topological conjugacy class contains all functions which share the same dynamics from a topological viewpoint. Two maps which are topologically conjugate cannot be distinguished topologically.

Recall that an iterated function $f^n : X \to X$ (from a space X to itself) is one which is obtained by composing another function $f : X \to X$ with itself a certain number of times. An orbit is a collection of points relating to the evolution of a dynamical system given a set of initial conditions.

Given an initial $x_0 \in X$, the orbit of x_0 is the set generated by repeatedly applying the function f to x_0. The orbit of x_0 under f is

$$\{x_0, x_1 = f(x_0), x_2 = f^2(x_0), ...\}$$

Lemma 2.6. *Given two topologically conjugate functions f, g, the orbits of f are mapped to the homeomorphic orbits of g through conjugation.* [5]

Proof. From above, we can write $g = hfh^{-1}$. Therefore, we can define the iterated function

$$g^n = hfh^{-1} \circ hfh^{-1} \circ hfh^{-1}...$$

Since function composition is associative, we obtain the relation

$$g^n = hf^nh^{-1}.$$

Therefore the orbit of $x_0 =$

$$\{x_0, x_1 = f(x_0), x_2 = f^2(x_0), ...\}$$

is mapped to the orbit of $y_0 = h(x_0) =$

$$\{y_0, y_1 = g(x_0), y_2 = g^2(x_0), ...\}$$

by topological conjugation. These two orbits are topologically equivalent under h, i.e. $y_k = h(x_k)$ for all $k \geq 0$. □

Topological conjugation implies that there are two routes for obtaining any $y_n \in Y$:

$$y_n = g^n(y_0) = g^n(h(x_0))$$

or

$$y_n = h(x_n) = h(f^n(x_0)).$$

See the diagram below for a more pictorial view of these routes.

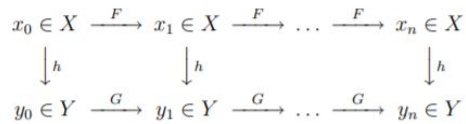


FIGURE 3. Topologically conjugate maps F, G via homeomorphism h sending $x_0 \to y_0$. [5]

2.4.2. *Topological properties.*

Lemma 2.7. *If f and g are topologically conjugate maps, then they have exactly the same topological properties.*

Lemma 2.8 (Converse)**.** *If two maps f and g do not have the same topological properties, then they are not topologically conjugate.*

Functions f and g being topologically conjugate means that they acquire identical topological properties including the same number of fixed points, orientation preservation and periodic orbits of the same stability type.

Example 2.9. Consider $f(x) = 2x$ and $g(x) = -2x$. The maps f and g are not topologically conjugate since f preserves orientation while g reverses orientation. This observation is sufficient to prove there is no topological conjugacy h for these two maps. [2]

Example 2.10. Consider $f(x) = 2x$ and $g(x) = \frac{1}{2}x$. The maps f and g are not topologically conjugate since $\lim_{n\to\infty} g^n(x) = 0 \in \mathbb{R}$ for all $x \in \mathbb{R}$ but $\lim_{n\to\infty} f^n(x) = \pm\infty \notin \mathbb{R}$ for all $x \neq 0$. [2]

Example 2.11. Consider $f(x) = 2x$ and $g(x) = 8x$. The maps f and g are topologically conjugate through the topological conjugacy $h : \mathbb{R} \to \mathbb{R},\ h(x) = x^3$. From this, we can determine that f and g possess the same topological properties, including the same number of fixed points. [2]

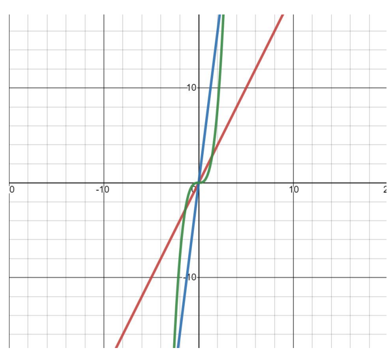


FIGURE 4. Topologically conjugate maps f (red), g (blue) via homeomorphism h (green).

Example 2.12. Consider the continuous maps $f, g : [0, 1] \to [0, 1]$ such that

$$f(x) = \begin{cases} 2x & \text{if } x < \frac{1}{2} \\ 2 - 2x & \text{if } x \geq \frac{1}{2} \end{cases}$$

and

$$g(x) = 4x(1 - x).$$

These two maps are topologically conjugate. [3]

Proof. By investigation, there exists a homeomorphism $h : [0, 1] \to [0, 1]$ where

$$h(x) = sin^2(\frac{\pi x}{2})$$

such that

$$h(f(x)) = g(h(x))$$

for all $x \in [0, 1]$.

$x < \frac{1}{2} \Rightarrow f(x) = 2x$. Then,

$$h(f(x)) = sin^2(\pi x).$$

Also,

$$g(h(x)) = 4sin^2(\frac{\pi x}{2})(1 - sin^2(\frac{\pi x}{2})).$$

Using trigonometric manipulation, we can express this as

$$g(h(x)) = 4sin^2(\frac{\pi x}{2})cos^2(\frac{\pi x}{2}).$$

$$\Rightarrow g(h(x)) = (2sin(\frac{\pi x}{2})cos(\frac{\pi x}{2}))^2.$$

$$\Rightarrow g(h(x)) = sin^2(\pi x).$$

So $f \sim g$ for $x < \frac{1}{2}$.

$x \geq \frac{1}{2} \Rightarrow f(x) = 2 - 2x$. Then,

$$h(f(x)) = sin^2(\frac{\pi}{2}(2 - 2x)).$$

$$\Rightarrow h(f(x)) = sin^2(\pi - \pi x)).$$

$$\Rightarrow h(f(x)) = sin^2(\pi x)).$$

So $f \sim g$ for $x \geq \frac{1}{2}$. Proving that f and g are topologically conjugate means that given the periodic orbits of g for example, we can trivially determine the periodic orbits of f through topological conjugation. We still need to ensure that h is a homeomorphism, i.e. a continuous bijective function with a continuous inverse.

- Firstly, we can quickly establish that h is onto or surjective, since

  $$h([0, 1]) = [0, 1],$$

  i.e. all values in the co-domain are obtained.
- Secondly, h is injective, since the function $h(x)$ is strictly increasing over $[0, 1]$. Thus, h is a bijection.
- Intuitively, we can establish that h is continuous, since the *sin* function is continuous over $[0, 1]$.

- There exists a continuous inverse, h^{-1}, where
$$h^{-1}(x) = \frac{2}{\pi} sin^{-1}(\sqrt{x})$$
which is continuous over $[0, 1]$. Therefore, h is a homeomorphism.

□

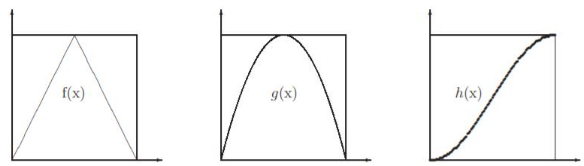


FIGURE 5. Plots of f, g, h. [3]

Example 2.13. Let $\mu > 0$. The map $f(x) = \mu x(1 - x)$ is topologically conjugate to a map $g(x) = x^2 + c$ for a suitable value of c.[6]

Proof. Let us assume the homeomorphism takes the form $h(x) = \alpha x + \beta$. Then h must satisfy the identity
$$h \circ f = g \circ h.$$
$$\Rightarrow \alpha\mu x(1 - x) + \beta = (\alpha x + \beta)^2 + c \forall\ x.$$
$$\Rightarrow \alpha\mu x - \alpha\mu x^2 + \beta = \alpha^2 x^2 + 2\alpha\beta x + \beta^2 + c.$$
$$\Rightarrow -\alpha\mu x^2 = \alpha^2 x^2,\ \alpha\mu x = 2\alpha\beta x,\ \beta = \beta^2 + c.$$
$$\Rightarrow -\mu = \alpha,\ \frac{\mu}{2} = \beta,\ \beta(1 - \beta) = c.$$
Therefore $f(x) = \mu x(1 - x)$ is topologically conjugate to $g(y) = y^2 + c$ if and only if $c = \frac{\mu}{2}(1 - \frac{\mu}{2})$. The topological conjugacy $h(x)$ is $-\mu x + \frac{\mu}{2}$. □

2.5. **Topological conjugacy of flows.**

Definition 2.14. Suppose, ϕ on X, and ψ on Y are flows, and $h : X \to Y$ as above.

ϕ being topologically semiconjugate to ψ means, by definition, that h is a surjection such that $h \circ \phi(x, t) = \psi \circ (h(x), t)$, for each $x \in X$, $t \in \mathbb{R}$.

ϕ and ψ are topologically conjugate if they are topologically semiconjugate and h is a homeomorphism.[8]

Example 2.15. The flow of $\frac{dx}{dt} = -x$ is topologically conjugate to the flow of $\frac{dy}{dt} = -2y$.[7]

Proof. Given the above differential equations we can easily solve both.
$$\frac{dx}{dt} = -x. \Rightarrow x(t) = x_0 e^{-t}$$
and
$$\frac{dy}{dt} = -2y. \Rightarrow y(t) = y_0 e^{-2t}$$
We find that the flows corresponding to these differential equations are $\phi(x, t) = xe^{-t}$ and $\psi(y, t) = ye^{-2t}$.

Let's try $h(x) = x^2$. Then we have

$$h(\phi(x,t)) = x^2 e^{-2t}.$$
$$\Rightarrow h(\phi(x,t)) = h(x)e^{-2t}.$$
$$\Rightarrow h(\phi(x,t)) = \psi(h(x),t).$$

$h(x) = x^2$ is a homeomorphism only for $x \geq 0$. Let's try $h(x) = -x^2$. Then we have

$$h(\phi(x,t)) = -x^2 e^{-2t}.$$
$$\Rightarrow h(\phi(x,t)) = h(x)e^{-2t}.$$
$$\Rightarrow h(\phi(x,t)) = \psi(h(x),t).$$

$h(x) = -x^2$ is a homeomorphism only for $x < 0$. Therefore, we have

$$h(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0. \end{cases}$$

By observation, h is a homeomorphism, since h is a continuous bijective function with a continuous inverse. So these flows are topologically conjugate. □

References

1. Wikipedia Contributors, *Topological conjugacy.* https://en.wikipedia.org/wiki/Topological$_c$*onjugacy*.
2. J.W. Robbin, *Topological conjugacy and structural stability for discrete dynamical systems*, Bulletin of the American Mathematical Society **78** (1972), no. 6, pp.925.
3. Makar Plakhotnyk, *Topological conjugation of one dimensional maps. (Draft book)*, (2016), pp.1.
4. Yuri A. Kuznetsov, *Conjugate maps.*, Scholarpedia **2** (2007), no.12, pp.5420.
5. Unknown Author, *Topological conjugacy - summary.* https://www.math.ubc.ca/ andrewr/620341/pdfs/tc$_s$*um.pdf*.
6. Unknown Author, *Dynamical systems and symbolic dynamics.* http://www.maths.qmw.ac.uk/ sb/cf$_c$*hapter4.pdf*.
7. Unknown Author, *Topological conjugacy and equivalence.* http://www.math.wsu.edu/math/faculty/schumaker/Math512/512F10Ch4B.pdf.
8. Zbigniew Leśniak, *On the topological conjugacy of Brouwer Flows.*, Bulletin of the Malaysian Mathematical Sciences Society, **42** (2019), pp.1639-1655.

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

THE HAUPTVERMUTUNG

THOMAS O'CONNOR, DARRAGH MCCRANN, AND ADAM MCDONAGH

(Communicated by David Futer)

Abstract. This paper aims to provide MA342 students with a deeper understanding of the Hauptvermutung, a conjecture that was at the heart of combinatorial and geometric topology for many years since the turn of the twentieth century, despite being proved false in 1961. We will state the Hauptvermutung, explain its significance and examine some of the attempts at proving and disproving the conjecture.

1. Introduction

The Hauptvermutung, or more formally, *Die Hauptvermutung der kombinatorischen Topologie* (the Main Conjecture of Combinatorial Topology), states that for two triangulations K, L of a topological space X, there exists subdivisions K', L' such that $K' = L'$. Originally put forward by Ernst Steinitz and Heinrich Tietze in 1908, the Hauptvermutung was formulated in an effort to prove that two homeomorphic spaces have the same Euler characteristic, that is, they are combinatorially equivalent. Despite being validated for low dimensional simplicial complexes and manifolds, a counterexample to the conjecture was presented by John Milnor in 1961 and eight years later, Robion Kirby and Laurent C. Siebenmann presented a general disproof for manifolds of dimension five or greater.


Figure 1. Heinrich Tietze (left) and Ernst Steinitz

2020 *Mathematics Subject Classification.* Primary 57Q25.

2. Useful Definitions

Simplicial Complex: A finite collection K of simplices in $\mathbb{R}^n$ is a simplicial complex if all faces of a simplex $\alpha \in K$ lie within K and, if two simplices in K intersect, then they do so in a common face lying in K.

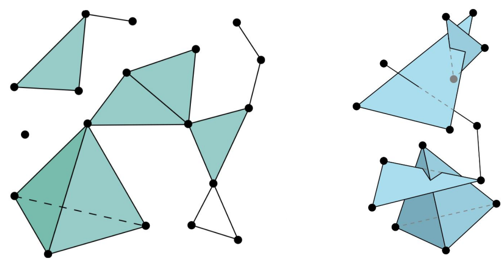


Figure 2. Simplicial (left) and non-simplicial complex

Let $|K|$ denote the topological realization of the simplicial complex K, which is K endowed with the usual Eulidean subspace topology. This will also be referred to as the polyhedron of K.

Homeomorphism: A continuous function $f : X \to Y$ is a homeomorphism if there exists a continuous function $g : Y \to X$ such that for all $x \in X$ and all $y \in Y$ $g(f(x)) = x$ and $f(g(y)) = y$.

Triangulation: A triangualtion of a topological space X consists of a simplicial complex K and a homeomorphism $h : |K| \to X$.

Euler Characteristic: The Euler characteristic of a simplicial complex K is defined as

$$\chi(K) = \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + ...$$

where α_i denotes the number of i-simplices of K.

Homotopy: Two maps $f, g : X \to Y$ are homotopic if there exists a homotopy

$$H : X \times [0,1] \to Y, \quad (x,t) \to H(x,t)$$

such that

$$H(x,0) = f(x)$$

and

$$H(x,1) = g(x).$$

For the purposes of this paper, the Hauptvermutung will be considered in two seperate cases: the Polyhedra Hauptvermutung and the Manifold Hauptvermutung, as in Raniki [1].

3. The Hauptvermutung for Polyhedra

The following preliminary definitions and theorems will be central to understanding the Polyhedra Hauptvermutung:

Subdivision: Let $|K|$ be the polyhedron of simplicial complex K. A simplicial complex K' is a subdivision of K if $|K'| = |K|$ and each simplex $\beta \in K'$ lies within some simplex $\alpha \in K$.

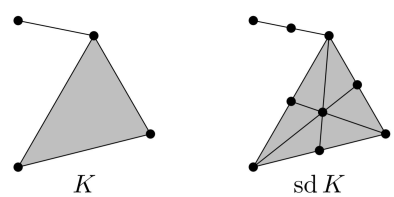


Figure 3. Barycentric Subdivision [2]

Figure 3 shows an example of barycentric subdivision which involves subdividing an n-simplex into $(n+1)!$ n-simplices by finding barycenters, or centroids. For example, the 2-simplex in the simplicial complex above is subdivided into six triangles, each consisting of a vertex at the centroid of the original triangle, at the midpoint of a side and at one of the original vertices.

Simplicial Map: The map $f : |K| \to |L|$, between simplicial complexes K,L, is simplicial if $f(\alpha)$ maps $\alpha \in K$ linearly onto a simplex of L. If f is bijective then it is a simplicial isomorphism.

Piecewise Linear: The map $f : |K| \to |L|$ is piecewise linear (PL) if there are subdivisions K' and L' such that $f : K' \to L'$ is simplicial. $|K|$ and $|L|$ are piecewise linearly homeomorphic if $K' \cong L'$.

Simplicial Approximation Theorem: *Every continuous map $f : |K| \to |L|$ between polyhedra is homotopic to the topological realization of a simplicial map $f' : K' \to L'$.*

The Simplicial Approximation Theorem states, therefore, that a continuous map between polyhedra is homotopic to a PL map. However, the Hauptvermutung is concerned with homeomorphisms of polyhedra and their subdivisions. It does not follow from the theorem above that that a homeomorphism of polyhedra is homotopic to a PL homeomorphism [1]. This is, in summary, what the Hauptvermutung for Polyhedra hypothesizes.

Polyhedra Hauptvermutung: *Every homeomorphism $f : |K| \to |L|$ between polyhedra is homotopic to the topological realization of a simplicial isomorphism $f' : K' \to L'$.*

Although the Polyhedra Hauptvermutung has been proven false in general, it holds for low dimensional polyhedra. In 1943, Papakyriakopoulos [3] proved the Hauptvermutung for all polyhedra of dimension 2. Five years later, Moïse [4] constructed a proof for the case of 3-dimensional manifolds. With such developments in corroborating the Hauptvermutung in low dimensional spaces, it appeared that topologists were homing in on a general ratification of the conjecture. In 1961, over fifty years since the Hauptvermutung was first put forward, however, John Milnor instigated a change in the general approach towards the conjecture when he illustrated a counterexample in higher dimensional polyhedra.

John Milnor: Born in 1931, John Milnor is an American mathematician renowned for his pioneering developments in differential topology, K-theory and dynamics. Currently Professor of Mathematics at Stony Brook University in New York, he was awarded a BA in Mathematics from Princeton in 1951 and a PhD in 1954.

Milnor was awarded the Fields Medal in 1962 for proving "that a 7-dimensional sphere can have several differential structures [which] led to the creation of the field of differential topology". He was also awarded the Wolf Prize (1989) and the Abel Prize (2011).

FIGURE 4. John Milnor

Milnor and the Hauptvermutung: With regards to the Hauptvermutung, Milnor found two simplicial complexes that are homeomorphic but have combinatorially distinct triangulations. Specifically, that there exists a homeomorphism $f : |K| \to |L|$ between the polyhedra of simplicial complexes K, L such that f is not homotopic to a PL homeomorphism.

Milnor's counterexample pertained to compact polyhedra alone and was, in his own words, "a rather pathological example, not about manifolds" [5]. However, it opened the door for further scrutiny of the Hauptvermutung in high dimensional manifolds. Indeed, one of Milnor's students, Laurent Siebenmann, along with Robion Kirby, was integral in proving that the Manifold Hauptvermutung is false in general.

3. The Hauptvermutung for Manifolds

Manifolds form another fundamental class of spaces studied in geometric and differential topology. In the context of the Hauptvermutung and for the purposes of this paper, we will focus on topological manifolds and in particular, a class of topological manifolds known as combinatorial or piecewise linear (PL) manifolds.

Topological and PL Manifolds: An m-dimensional topological manifold is a space that looks locally like m-dimensional Euclidean space. A piecewise linear manifold is a topological manifold equipped with a piecewise linear structure.

At the heart of the Hauptvermutung for manifolds lies the question whether or not every topological manifolds admits a PL structure.

Manifold Hauptvermutung: *Every homeomorphism $f : |K| \to |L|$ of the polyhedra of compact m-dimensional PL manifolds is homotopic to a PL homeomorphism.*

Closely related to the Manifold Hauptvermutung, the Combinatorial Triangulation Conjecture states that every compact m-dimensional topological manifold can be triangulated by a PL manifold. Both conjectures have been verified for low-dimensional manifolds. In 1925, Tibor Radó proved that both conjectures hold for manifolds of dimension $m = 2$. Edwin E. Moïse then corroborated the conjectures for 3-dimensional manifolds in 1952.

Kirby and Siebenmann: In 1969, however, Robion Kirby and Laurent Siebenmann proved that both the Manifold Hauptvermutung and the Combinatorial Triangulation Conjecture are false in general. In particular, that for all $m \geq 5$, there exists topological m-dimensional manifolds that do not posses PL structures, that is, they do not have a combinatorial triangulation.

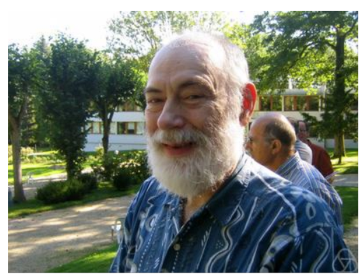


Figure 5. Robion Kirby (left) and Laurent Siebenmann

Developments in surgery theory, in particular the surgery classification of PL structures on high-dimensional tori, were central in Kirby and Siebenmann's proof. They also utilized the Rohkhlin topological invariant to construct their own invariant, the Kirby-Siebenmann invariant, which detects when the Hauptvermutung and the Combinatoiral Triangulation Conjecture fail [1].

It is worth noting that by 1969, the Hauptvermutung had been proven to hold for manifolds of dimension 2 and 3 but shown not to be true for any dimension 5 or greater. "Dimension four is very special in topology", according to Manolescu [6], which may explain why it was not until 1982 that a counterexample to the Manifold Hauptvermutung for dimension 4 was presented when Michael Freedman discovered the E_8 manifold. E_8 has no PL structure and in fact, cannot be triangulated at all, by the Casson's invariant [1].

4. Conclusion

In light of the above developments in disproving the Hauptvermutung, Milnor's counterexample and the Kirby-Siebenmann Theorem in particular, topologists turned to homotopy equivalence in order to prove the topological invariance of the Euler characteristic.

Although the Hauptvermutung is now known to be false in general, the conjecture prompted many significant developments in geometric, combinatorial and differential topology. The existence of Milnor's exotic spheres and Freedman's exotic $\mathbb{R}^4$, for example, marked a significant shift in the mathematical understanding of higher dimensional spaces which contain various differentiable structures. Such findings were vital in the development of many other theorems and conjectures in topology such as the Gereralized Poincaré Conjecture.

> "*A felicitous but unproved conjecture may be of much more consequence for mathematics than the proof of many a respectable theorem*"
>
> —— Atle Selberg [7]

5. References

[1] A.A.Ranicki (ed.). *The Hauptvermutung Book.* Springer, the Netherlands (1996).

[2] J, Loher. *Simplicial Approximation.* Seminar, 25 November 2015.

[3] C, Papakyriakopoulos. *A new proof of the invariance of the homology groups of a complex.* Bull.Soc.Math. Gréce 22 (1943), 1–154.

[4] E, Moïse *Affine structures in 3-manifolds.* Ann. of Maths. 58 (1953), 458–480.

[5] M. Raussen and C. Skau. *Interview with John Milnor.* 25 May 2011, Oslo. In: Notices of the American Mathematical Society, Vol.59(3) (2011), 400–408.

[6] C. Manolescu. *Triangulations of Manifolds.* Notices of the ICCM. International Press (2014), 21–23.

[7] A. Selberg. *Reflections around the Ramanujan Centenary.* Ramanujan: Essays and Surveys. American Mathematical Society (2001), 203–214.

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

SMOOTH MANIFOLDS AND LIE GROUPS

(Communicated by Michael McGloin and Oisin McGloin)

ABSTRACT. The aim of this project is to provide an overview of what a manifold is and to look at some properties they possess. We defined a manifold and described what is meant for it to have a smooth structure. We then proved that certain objects are manifolds and investigated diffeomorphisms between different manifolds. We also discuss Lie groups and prove that S^2 is not a Lie group using the Hairy Ball Theorem

1. MANIFOLDS

Firstly we'll look at the formal definition of a manifold and then explain some of the terms. A manifold is a second countable Hausdorff topological space M equipped with an atlas A = (U , φ). The atlas is a collection of what are called charts and maps between these charts. These charts are defined as follows:

Let $p \in M$ and U_p be an open set around p such that $\exists$ a homeomorphism $\varphi_p : U_p \mapsto \varphi_p(U_p) \subseteq R^d$. (U_p, φ_p) is called a chart and must satisfy the following conditions :

$$M = \bigcup_i U_i,$$

There exists transition maps between these charts such that :

$$\psi_{xy} : \phi_x(U_x \cap U_y) \mapsto \phi_y(U_x \cap U_y),$$
$$\text{given by } \psi_{xy} = \phi_y \circ \phi_x^{-1}$$

If these transitions maps are continuous and infinitely differentiable the manifold is called a smooth manifold. More informally a manifold is some space that locally resembles Euclidean Space.

Example 1.1. The unit circle S^1 is a smooth manifold.

Proof. Let $S^1 = \{x, y \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ denote the unit circle with the subspace topology inherited from $\mathbb{R}^2$ with the standard topology. Since $\mathbb{R}^2$ is Hausdorff so is any subspace of it.Now let's construct charts for S^1.
Let $U_1 = \{p \in S^1 : y > 0\}$, $\phi_1(p) = x$, $\phi_1^{-1}(p) = (x, \sqrt{1 - x^2})$
Let $U_2 = \{p \in S^1 : y < 0\}$ $\phi_2(p) = x$, $\phi_2^{-1}(p) = (x, -\sqrt{1 - x^2})$
Let $U_3 = \{p \in S^1 : x < 0\}$ $\phi_3(p) = y$, $\phi_3^{-1}(p) = (-\sqrt{1 - y^2}, y)$
Let $U_4 = \{p \in S^1 : x > 0\}$ $\phi_4(p) = y$, $\phi_1^{-1}(p) = (\sqrt{1 - y^2}, y)$

2010 *Mathematics Subject Classification.* Primary .

All of these ϕ_i are bijective and continuous on their domains and $S^1 = \bigcup_i U_i$.
$\therefore \forall p \in S^1 \ \exists$ a homeomorphism that maps p to $\mathbb{R}^1$ so S^1 is locally Euclidean of dimension 1.

Furthermore if we consider the transition maps between our charts we first see that $U_1 \cap U_2 = \emptyset$ and $U_3 \cap U_4 = \emptyset$.

Also the transition functions :

$$\psi_{1,3} = \psi_{1,4} = \phi_3\phi_1^{-1} = \sqrt{1-t^2}$$
$$\psi_{2,3} = \psi_{2,4} = \phi_3\phi_2^{-1} = -\sqrt{1-t^2}$$

Are continuous and smooth on their respective domains.
$\therefore$ S^1 is a smooth 1-manifold.

□

Now we will look at an important example of a smooth manifold.

Example 1.2. The set of $GL_n(\mathbb{R})$ of real n*n matrices with non-zero determinant is a smooth manifold.

Proof. We can map each entry of a n*n matrix trivially to a real number and consider the n*n-tuple $\in \mathbb{R}^{n^2}$ formed from this map. This takes each element of $GL_n(\mathbb{R})$ to a point in $\in \mathbb{R}^{n^2}$ and the map is trivially smooth. Since $\mathbb{R}^{n^2}$ is Hausdorff and second countable this forms a smooth manifold. □

Some more examples of manifolds include : The Torus, The Real Projective Plane RP^n and SO(n) the set of orthogonal n*n matrices with det = 1.

Now we are going to look at some definitions that will help us work with manifolds.

Definition 1.3 (Diffeomorphism)**.** Given two manifolds M and N, a differential map $f : M \mapsto N$ is called a diffeomorphism if it is bijective and its inverse $f^{-1} : N \mapsto M$ is differentiable.
Two manifolds M and N are diffeomorphic if there exists a diffeomorphism between them.

The role of a diffeomorphism is to preserve the differentiable structure of a smooth manifold. For example if $\phi : M \mapsto N$ is smooth then $g : N \mapsto \mathbb{R}$ is smooth if and only if $g \circ \phi : M \mapsto mathbbR$ is smooth.

Definition 1.4 (Smooth Submanifold)**.** Let M be a smooth manifold of dimension m with N ⊆ M. We call N a smooth n-dimensional submanifold of M if ∀ c $\in N$ ∃ a smooth chart, (U, ϕ) in M, such that c ∈ U and $\phi(N \cap U) = \mathbb{R}^n \cap \phi(U)$ where $\mathbb{R}^n$ is embedded into $\mathbb{R}^m$ as the subspace $\{x_{n+1} = 0, ..., x_m = 0\}$.

Intuitively a submanifold can be thought of as a subset of some manifold which itself has the structure of a manifold.

We will look at an example of a diffeomorphism between SO(2), which is the group of 2x2 matrices with determinant 1, and S^1 the circle.

Example 1.5. To show that SO(2) is diffeomorphic to S^1 we need to find a diffeomorphism between them. We can represent the elements of SO(2) by the following matrix :

$$A = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

where the determinant equals 1.

For any matrix A $\in$ SO(2) we can consider the map :

$$\text{SO}(2) \mapsto S^1 \text{ by}$$
$$\text{A} \mapsto e^{i\theta}.$$

Since $e^{i\theta}$ is bijective and its inverse is differentiable, we have found a diffeomorphism between SO(2) and S^1, so SO(2) is diffeomorphic to S^1.

We can also show that S^1 is a submanifold of $\mathbb{R}^2$ as follows:

Example 1.6. We want to show that S^1 given by $x^2 + y^2 = 1$ is a submanifold of $\mathbb{R}^2$. Consider a point (x,y) in our circle. If $y > 0$, then near that point, the graph of the circle can be given by $\sqrt{1-x^2}$. We know this is smooth since our x only ranges from (-1,1). Similarly for $y < 0$ we can represent our circle by $-\sqrt{1-x^2}$. This is again smooth by the same logic as before. For y = 0, we just get x = $\sqrt{1-y^2}$ or x = $\sqrt{1-y^2}$. Here we have constructed smooth charts and $\phi(S^1 \cap U) = \mathbb{R} \cap \phi(U)$. $\therefore$ The unit circle is a submanifold of $\mathbb{R}^2$.

.

2. Lie Groups

Earlier we seen that $GL_n(\mathbb{R})$ and SO(2) were both manifolds. However these sets of matrices also satisfy the group axioms under matrix multiplication. This leads us to a very important class of manifolds called Lie Groups.

Definition 2.1 (Lie Groups)**.** Lie groups are groups that are also manifolds such that the multiplication maps :

$$m : G \times G \mapsto G \ , \ (g, h) \mapsto gh$$

and

$$i : G \mapsto G \ , \ g \mapsto g^{-1}$$

are infinitely differentiable (i.e smooth) .

Lie groups allow us to talk about continuous symmetry and arise in many different areas of both pure maths and physics. So far we have seen that S^1 is a Lie group but what about S^2? In order to answer this question we will have to introduce some more technology.

Definition 2.2 (The Tangent Space)**.** Let M be a smooth Manifold and $C^\infty(M)$ denote the space of infinitely differentiable functions on M. For a point $x \in M$ we define a derivation at that point to be a linear map $D : C^\infty \mapsto \mathbb{R}$ that satifies the Liebniz identity (essentially the product rule)

$$\forall g, f \in C^\infty \colon\ D(f,g) = D(f) * g + f * D(g)$$

We can now define addition and scalar multiplication at x by :

$$(D_1 + D_2)(f) = D_1(f) = D_2(f)$$
$$(\lambda D)(f) = \lambda(D(f))$$

From this we obtain a vector space denoted T_xM the tangent space of M at the point x.

For some intuition we can consider the unit sphere in $\mathbb{R}^3$. The tangent space T_xS^2 is the set of all tangent vectors on S^2 at a point x where tangent vectors are defined as follows :
$v \in \mathbb{R}^3$ is a tangent vector of S^2 at x if there is a smooth curve $\gamma : \mathbb{R} \mapsto S^2$ such that :

$$\gamma(0) = x$$
$$\dot{\gamma}(0) = v$$

The tangent space at a given point of our unit sphere will be a plane that touches the sphere only at that point.
A vector field on a manifold is an assignment of a tangent vector to each point of our manifold. It is said to be non-vanishing if it is non-zero at every point.
Now we will state (without proof) at an important theorem in algebraic topology.

Theorem 2.3 (The Hairy Ball Theorem)**.** *There is no non-vanishing tangent vector field on* S^2

Theorem 2.4. *We will now prove that if G is a Lie group there exists a non-vanishing tangent vector field.*

Proof. Let v be a non-zero vector in T_eG where $e \in G$ is the identity element. For any $g \in G$ consider the left multiplication $\phi_g : G \mapsto G$, $x \mapsto gx$ which is a diffeomorphism of G since G is a Lie group. This induces the isomorphism $d\phi_g : T_eG \mapsto T_gG$, which maps v to another non-zero vector $\phi_g(v) \in T_gG$ since v itself is non-zero. Then $\{\phi_g(v)\}$ defines a smooth tangent vector field, which is non-zero anywhere. □

So from this we can see that if S^2 where to be a Lie group it would need to have a non-vanishing tangent vector field which contradicts the Hairy Ball Theorem.
$\therefore S^2$ is not a Lie Group.

3. Bibliography

S^2 is not a Lie Group, Seewoo Lee, 24/08/2018, https://math.berkeley.edu/ seewoo5/S2notLiegrp.pdf

Introduction to Smooth Manifolds, John Lee, 2012

WHYB Math Introducing Manifolds, https://www.youtube.com/watch?v=GqRoiZgd6N8t=159s

Introduction to Lie Groups and Lie Algebras, Alexander Kirillov, Jr

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

AYASDI, TOPOLOGICAL DATA ANALYSIS, AND ITS APPLICATIONS

COLM ÓG CONNEELY, DARRAGH FLANNERY, AND JONATHAN HESTER

(Communicated by David Futer)

Abstract. Despite the well known quote by Solzhenitsyn; "Topology! The stratosphere of human thought! In the twentyfourth century it might just possibly be of use to somebody, but for the present ...", Gunnar Carlsson and Ayasdi have found a practical use for topology in the form of Topological Data Analysis(TDA). This paper explores Topological Data Analysis, its foundation, and its benefits. This paper describes and illustrates how this new way of looking at data can reap real world benefits. We discuss the topological properties that make Topological Data Analysis possible, and how TDA provides a framework for machine learning which helps form a better understanding of the data being analysed. We discuss Ayasdi, and their groundbreaking approach to data analytics. We also explore the real world solutions offered by Ayasdi based off their use of Topological Data Analysis.

1. Ayasdi

Ayasdi is an American company which was founded in 2008 by Gunnar Carlsson, a Stanford mathematician [1]. The company offers offers data analysis based on topology with vast applications ranging from disease research to investing to fraud detection. "We need better algorithms to ask the right questions", this is the premise on which Ayasdi works. Topological data analysis can be used to identify patterns in big data, and Ayasdi uses this to provide automated data analytics at scale. Ayasdi's automated approach to machine learning has been found to be two to five times more efficient than alternative approaches to big data analytics. Having solved a range of problems ranging from polio/refugees to disease recovery, genomics, spinal cord injury/traumatic brain injury, asthma, diabetes, and even predicting earthquakes, Ayasdi has created a buzz around their products and have attracted significant media coverage.

2. Gunnar Carlsson

Gunnar E. Carlsson is an American mathematician, working in algebraic topology. He is known for his work on the Segal conjecture, and for his work on applied algebraic topology, especially topological data analysis. He is a Professor Emeritus in the Department of Mathematics at Stanford University. Carlsson was born in

Received by the editors 30/05/2021.
2020 *Mathematics Subject Classification.* Primary,62R40.
1

Sweden and was educated in the United States. He received a Ph.D. from Stanford University in 1976, with a dissertation written under the supervision of R. J. Milgram. He was a Dickson Assistant Professor at the University of Chicago (1976-1978) and Professor at the University of California, San Diego (1978–86), Princeton University (1986-1991), and Stanford University (1991–2015). Carlsson's work within topology encompasses three areas: • Equivariant methods in homotopy theory • Algebraic K-theory • Applied and computational topology. In 2008, Carlsson cofounded Ayasdi, a predictive technology based on big data, machine learning and artificial intelligence, that uses topological analysis in its work. [2]

3. Subgroups of Topology

Topology has many different subfields: • General Topology. General topology normally considers local properties of spaces, and is closely related to analysis. It generalizes the concept of continuity to define topological spaces, in which limits of sequences can be considered. Sometimes distances can be defined in these spaces, in which case they are called metric spaces; sometimes no concept of distance makes sense. • Combinatorial Topology. Combinatorial topology considers the global properties of spaces, built up from a network of vertices, edges, and faces. This is the oldest branch of topology, and dates back to Euler. It has been shown that topologically equivalent spaces have the same numerical invariant, which we now call the Euler characteristic. This is the number (V - E + F), where V, E, and F are the number of vertices, edges, and faces of an object. For example, a tetrahedron and a cube are topologically equivalent to a sphere, and any "triangulation" of a sphere will have an Euler characteristic of 2. • Algebraic Topology. Algebraic topology also considers the global properties of spaces, and uses algebraic objects such as groups and rings to answer topological questions. Algebraic topology converts a topological problem into an algebraic problem that is hopefully easier to solve. For example, a group called a homology group can be associated to each space, and the torus and the Klein bottle can be distinguished from each other because they have different homology groups. Algebraic topology sometimes uses the combinatorial structure of a space to calculate the various groups associated to that space. • Differential Topology. Differential topology considers spaces with some kind of smoothness associated to each point. In this case, the square and the circle would not be smoothly (or differentiably) equivalent to each other. Differential topology is useful for studying properties of vector fields, such as a magnetic or electric fields.

4. Topological Data Analysis

Topology is the branch of mathematics associated with the study of shapes, how to 'measure' such shapes, and how properties are preserved under continuous deformations. Topology is incredibly useful in the analysis of complex, high dimensional data sets. Topology can allow us to 'measure' shape related properties in the data, like the presence of loops, also it provides a means for creating compressed representations of the data. It is topology that allows these compressed representations to retain features and reflect relationships of points in the data set.

---

[2]

5. Three Properties of Topological Analysis

In reality, topology is the study of shape from a specific perspective. It cannot tell the difference between a perfectly round circle and a circle that has been "squashed" into an ellipse, for example. Topological analysis has three key qualities, all of which are useful in a variety of situations.The three properties include coordinate invariance, deformation invariance and compressed representations.

5.1. **Coordinate Invariance.** Topology does not examine features of shapes that are dependent on the set of coordinates chosen, according to this property.This property is useful in data analysis since we frequently change data by applying various transformations to the entries in a data matrix, which results in a change of coordinates. Only translations and scaling are involved in very simple transformations, such as the conversion of Celsius temperatures to Fahrenheit or Kelvin. Complex transformations, such as three-dimensional rotations, are frequently used to illustrate a data set's underlying properties. The goal is to investigate properties that do not change as a result of such coordinate shifts. Another example of this property is in the field of gene expression microarray studies, where different technologies may be used to study the same phenomenon, such as specific types of cancer, but produce outputs with completely different coordinate systems, resulting from different gene or gene family choices as coordinates.

5.2. **Deformation Invariance.** When we stretch or deform a geometric shape, topological features remain intact, according to this property. Consider the difficulty of distinguishing between letters as an example. Independent of the font the letters are written in, the angle from which they are viewed, and even the curvature of the surface on which the letters are drawn, the human visual system is capable of robustly distinguishing the differences that distinguish between a letter "Y" and a letter "Z". The various font or aspect choices can be thought of as deformations of the letters' underlying geometry. When applying complex transformations to a data collection, such as log-log transformations, this kind of robustness comes in handy. For example, taking a perfectly round circle and applying a log-log transformation yields the shape below.

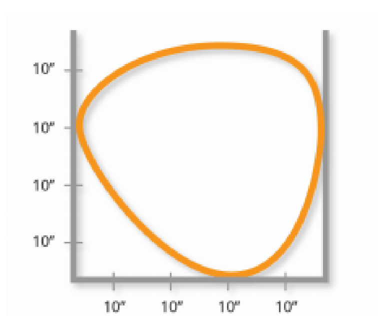


Figure 1. Applying a log-log transformation to a perfectly round circle

5.3. **Compressed Representations.** Finally, the last important property of topological methods is that they produce compressed representations of shapes. Consider the circle, which has an infinite number of points and infinite number of pairwise distances that define the shape. We can get a simple depiction of the fundamental "loopy" feature of the circle by utilizing a hexagon if we are willing to forgo a little detail, such as the curvature of the arc. This is enormously helpful in interpreting the characteristics of huge, complex data sets. In this scenario, the data set is made up of millions of points that have a similarity relationship. All of these relationships are encoded in the compressed representation as a topological network or complex, such as the hexagon.

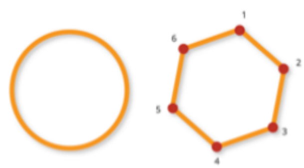


Figure 2. A simple depiction of the fundamental "loopy" feature of the circle by utilizing a hexagon

6. Topological Networks: The Future of Visualising Data

The concept of visualizing data is one that appeals to many people. It allows us to use our visual abilities to gain information and comprehension more quickly than we might by using query or algebraic approaches to examine the data. Data sets can be visualized in a variety of ways. Histograms, pie charts, bar graphs, heat maps and scatterplots are appealing ways of displaying and discovering data, all of which can provide valuable information.

The main distinction between these methodologies and Topological Data Analysis (TDA) is that TDA uses a topological network to interact with and represent both structured and unstructured data. Rather than providing a visual depiction of the behaviour of one or two of the variables characterizing the data set, a topological network gives a map of all the points in the data set, so that neighboring points are more similar than distant points. The network resembles a geographic map and serves the similar purpose in comprehending the "landscape" of the data.

A topological network depicting the Miller-Reaven data set, a well-known early diabetes data set, is shown below. TDA generated this network automatically, which cleanly divides the data set into three categories, each represented by a "flare". These correlate to well-known groupings of clinical outcomes: healthy patients are in the upper right blue flare, pre-diabetics are in the lower center blue flare, and overt diabetic patients are in the top left red flare. The colouring is based on glucose levels, indicating that overt diabetics have high blood glucose levels. The topological network is built automatically, and it clarifies the structure of the data set without querying it or without the need to perform algebraic analysis on a subset of variables.

If you work with other types of visualizations, you'll have to do a lot of manual work to identify this structure. To discover that these three groups exist, one

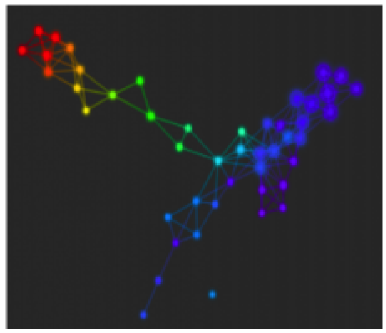


FIGURE 3. Miller-Reaven data set

would have to work with histograms and scatterplots of results for blood glucose and insulin response.

But perhaps more importantly, the network not only visualizes the data, but also provides an interactive paradigm for engaging with it. Variables utilized in the network's design, as well as information, can be utilized to colour the network. If you had a data set of diabetic patients, you could colour the nodes according to whether they had type I diabetes or not. Furthermore, any element of the network (and hence any component of the data collection) can be selected for further research and analysis of the fine grain structure within the data. After interrogating the data in this way, one can learn what distinguishes various subgroups and then look for statistically significant traits that separate each group from the rest of the data set.

This means that the topological network is simple to probe, allowing one to deduce the true meaning of the data by examining a compressed version of the data set that retains all of the subtle properties. As each node in this representation corresponds to several data points, the number of nodes is frequently substantially lower than the number of data points.Each node contains data points that are similar to one another in some way. As a result, the network provides much more than a static representation. Rather, it acts as a workbench for searching and evaluating data without the need for algebraic computations or database queries. It enables one to directly comprehend the data's general arrangement.[3]

7. Real Life Applications of Topology and Topological Analysis

Topology is the only major branch of modern mathematics that wasn't anticipated by the ancient mathematicians. Throughout most of its history, topology has been regarded as strictly abstract mathematics, without applications. However, illustrating Wigner's principle of "the unreasonable effectiveness of mathematics in the natural sciences," topology is now beginning to come up in our understanding of many different real world phenomena. Topology is applicable and useful in many facets of our lives. Some of the areas in which it is being used are:

7.0.1. *Biology.* Topology has been used to study various biological systems including molecules and nanostructure]). In particular, circuit topology and knot theory have been extensively applied to classify and compare the topology of folded proteins and nucleic acids.

[3]

7.0.2. *Computer Science.* Topological data analysis uses techniques from algebraic topology to determine the large scale structure of a set.

7.0.3. *Physics.* Topology is relevant to physics in areas such as condensed matter physics, quantum field theory and physical cosmology. In cosmology, topology can be used to describe the overall shape of the universe. This area of research is commonly known as spacetime topology.

8. How Ayasdi Uses Topological Data Analysis

"Gunnar Carlsson came up with the idea that you can drive understanding from very complex data sets by using the principles of topology or principle of shape," said Patrick Rogers, chief marketing officer at Ayasdi.

Traditional data analysis relies on statistics. Ayasdi offers data analysis based on topology with applications ranging from disease research to investing to fraud detection. Ayasdi works on the premise that "We need better algorithms to ask the right questions". That is, people exploiting data may not always know the right questions to ask, and advanced mathematics offer solutions to interpret large amounts of data starting with the data itself. The solutions developed rely on topological data analysis. Topology studies geometric properties of data, and TDA looks at the topology of data sets to recognize patterns and analyze the data. However, what is clear is that traditional data analysis has always used statistical tools to work with large amounts of data. In this radically different approach, variables represent coordinates of a data point in an n-dimensional set, and geometrical analysis is used to analyze data, and due to this fundamental difference in approach, different analysis and results become possible.

9. Topological Data Analysis As Framework For Machine Learning

Machine learning essentially provides us with a collection of techniques to help understand data,including methods for visualization, prediction, categorization and other techniques to make better sense of data. Generally, visualization techniques come in the form of scatterplot methods, where the data points are projected on two or three dimensions. Topological Data Analysis does not represent data in this way, rather is represents the data as a topological network. A topological network is produced by grouping similar data points into nodes, and if those nodes have a data point in common, they are joined by an edge. Since each node can represent multiple different data points, the network provides a compressed representation of high dimensional data. Other network representations methods do not allow for this compression and produce networks which are complex and can be difficult to interpret. Topological networks can be produced from machine learning techniques, therefore its possible to represent scatterplots as networks so that it is easier to understand and interact with the data being analysed[4]. These networks effectively output machine learning methods in a way which highlights the important segments of data. It is shown above how a scatter plot is represented as a network, and then further represented as a topological network.

4

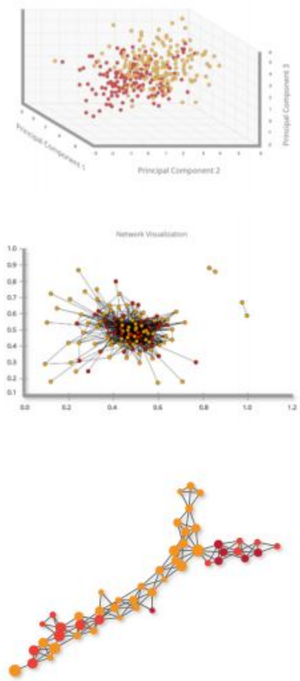


Figure 4. Topological Network Representation

10. Real World Solutions from Ayasdi

10.1. **Financial Crime and Fraud.** SensaAML is a leading AI produt offered by Ayasdi which is designed to detect financial crime, fraud and cyber-crime. Combining AI and machine learning, SensaAML SensaAML better analyses data and is proven to reduce the number of false positives, as well as being being able to discover other anomilies in the data. At the heart of the SensaAML technology is the ability to determine whether behaviour is suspicious or not, the software can find the optimal balance between signal and noise in the data. SensaAML is able to find and predict criminal activity in the data by identifying patterns that other softwares can't. The solution provides a map of all criminal behaviour, and is capable of providing the criminals exact coordinates on the map[5].

10.2. **Risk, Liquidity and Profitability.** Ayasdi's AI services can provide deep insights for financial institutions, highlighting behaviour and activity that previously could not be identified. It helps firms better understand customers, and how their behaviour impacts the firm's profitabilty, risk and operating model. Symphony AyasdiAI is capable of early identification of credit risk, the AI can allow banks to create a 90-day forward look into credit risk, and there is a 97 percent accuracy associated with this prediction. Historically, poor risk prediction models generally have resulted in too much capital being assigned to regulatory capital allocation, Ayasdi's AI solution instead focus on discovering opportunities to release

5

cash from corporate balances while remaining within optimal liquidity levels. After implementing AyasdiAI's solution, one bank was able to release 7 billion dollars of reserved money that was then invested and generated 110 Million in profit[6].

10.3. **Customer Intelligence.** True customer intelligence is essential to get a clear understanding of customers behaviour and actions. Customer intelligence helps to understand what to expect from customers both in the present and in the future. Ayasdi provides a simple, powerful way for institutions to optimise customer relations and customer retention. Ayasdi's solution uses current behaviours to discover new behaviours and can identify indicators of behaviour trajectories that are often missed by traditional data analytics methods[7]. Ayasdi can provide clients with early signals of changes in customer sentiment, and product relevance. Ayasdi's ability to predict such changes provides clients with a competitive edge.

10.4. **Healthcare.** Ayasdi's software also is of great use in healthcare. It solves a problem many physicians and healthcare organizations have shared and struggled with for years. "One of health cares top challenges, and it's true for all hospitals, is reducing variation in clinical care. When they have a patient come in for a particular procedure, it can be wildly variable in terms of patient outcomes, readmittance rates, cost of the procedure", said Patrick Rogers, chief marketing officer at Ayasdi. Ayasdi's benefits to healthcare can be seen through the company Mercy. Mercy, based in St. Louis, is one of the top five health systems in the US, as of 2016. [8] The managers of Mercy found that it had significant variation around certain procedures and wanted to develop best practices for these procedures based on their data. They made a decision to adopt the Ayasdi Care platform, which correlated and analyzed Mercy's electronic medical record and financial documentation. This included all information related to treatments , procedures , drugs administered, length of stay, and costs per patient. Specifically, the program incorporated the following elements: Artificial intelligence and data-based methods that allowed the company to carry out machine learning, statistics, and geometric algorithms. An application unique is its ability to accelerate the development of care models by drawing patient and clinical data directly from the health networks' integrated systems of record – not just benchmark data. A deep dive on cost and quality and care variations in knee replacements. Ayasdi's work with Mercy seems to be of great success as the company earned second place in the 2017 Healthcare Informatics Innovator Awards. [9]

References

[1] Christina Farr, *A cure for cancer? This 'big data' startup says it can deliver* (2013). https://venturebeat.com/2013/01/16/ayasdi/#1FPlL7mgRXdxYWOX.99

[2] En.wikipedia.org, *Gunnar Carlsson - Wikipedia* (n.d.) https://en.wikipedia.org/wiki/Gunnar_Carlsson

[3] Ayasdi.com, *Topology & Topological Data Analysis* (n.d.) https://www.ayasdi.com/wp-content/uploads/2015/02/Topology_and_Topological_Data_Analysis.pdf

[4] ———, . *Introduction to Topological Data Analysis* (n.d.) https://www.ayasdi.com/social/TDAintroduction/

6

7

8

9

[5] ———, *Financial Crime and Fraud* (n.d.)
https://www.ayasdi.com/solutions/financial-crime-fraud/
[6] ———, *AI-Powered Asset and Liability Management* (n.d.)
https://www.ayasdi.com/solutions/alm-risk-liquidity-and-profitability/
[7] ———, *Real KYX Intelligence: Transform Customer Experience with AI* (n.d.)
https://www.ayasdi.com/solutions/real-customer-intelligence/
[8] Kristan Lee, *Ayasdi uses topological data analysis to find best care paths* (n.d.)
https://www.advisorperspectives.com/articles/2020/02/03/james-montier-get-out-of-u-s-equities-and-into-emerging-markets/
[9] Prnewswire.com, . *Ayasdi Congratulates Mercy on Winning the Healthcare Informatics Innovator Award* (n.d.)
https://www.prnewswire.com/news-releases/ayasdi-congratulates-mercy-on-winning-the-healthcare-informatics-innovator-award-300401421.html

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

ON THE MAPPER CLUSTERING ALGORITHM

GRIFFEN SMALL AND WILLIAM LEACY

(Communicated by Julie Bergner)

ABSTRACT. In this article, we outline the Mapper clustering algorithm introduced by Singh, Mémoli and Carlsson in [1]. We also present computational results demonstrating the application of the algorithm to clinical data using the TDAmapper package for **R**.

1. Introduction

In applied mathematics, *topological data analysis* (TDA) is an approach to the analysis of datasets using techniques from topology. In many cases, particularly in biology, data coming from real-world applications is high-dimensional, incomplete and noisy, which severely restricts our ability to visualize it. Our ability to analyse such data, both in terms of quantity and the nature of the data, is also restricted. TDA aims to uncover, understand and exploit the topological and geometric structure underlying such data.

In [1], Singh, Mémoli and Carlsson propose a method which can be used to reduce high-dimensional datasets into *simplicial complexes* with far fewer points which can capture topological and geometric information. The method, which the authors refer to as Mapper, is based on *clustering* and the use of *filter functions* defined on the data in question.

2. Topological Preliminaries

The Mapper clustering algorithm is motivated by well-known constructions in topology which, in the interest of completeness, we now outline.

Definition 2.1. A *simplicial complex* $(V, \mathcal{K})$ consists of a finite set V and a family $\mathcal{K}$ of non-empty subsets of V called *simplices* with the following properties:

(1) $\{v\} \in \mathcal{K}$ for every $v \in V$;
(2) every non-empty subset of a simplex is itself a simplex.

Remark 2.2. We call the elements $v \in V$ *vertices*, and we call the simplex $\sigma \in \mathcal{K}$ an *n-simplex* if σ consists of $n + 1$ vertices. Moreover, we call 0-simplices *vertices*, 1-simplices *edges* and 2-simplices *triangles*.

Received by the editors January 1, 2021.
2020 *Mathematics Subject Classification.* Primary 55N31; Secondary 55U10.
Key words and phrases. Topological data analysis, Mapper, Simplicial Complex, Nerve.

Example 2.3. The pair $(V, \mathcal{K})$ with

$$V = \{1, 2, 3, 4\},$$
$$\mathcal{K} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$$

is an example of a simplicial complex. Figure 1 displays this simplicial complex as a graph.

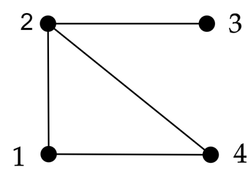


FIGURE 1. Graphical representation of the simplicial complex from Example 2.3.

Intuitively, a simplicial complex structure on a space is an expression of the space as a union of vertices, edges, triangles and higher dimensional analogues. Simplicial complexes provide a particularly simple way to describe certain topological spaces. For this reason, one often attempts to "approximate" topological spaces by simplicial complexes.

Let X be a topological space. There are numerous ways of constructing simplicial complexes from X together with additional data attached to X. In this article, we focus on the *nerve* of an *open cover* of X. Recall that a collection of open sets $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ is an *open cover* of a topological space X if $X = \bigcup_{i=1}^{n} u_i$.

Definition 2.4. Let $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ be a finite collection of sets. The *nerve* $N\mathcal{U}$ is a simplicial complex $(V, \mathcal{K})$ with $V = \{u_1, u_2, \ldots, u_n\}$ with a d-simplex $\sigma = \{u_{i_0}, u_{i_1}, \ldots, u_{i_d}\} \in \mathcal{K}$ whenever $u_{i_0} \cap u_{i_1} \cap \cdots \cap u_{i_n} \neq \emptyset$.

Remark 2.5. This is an extremely useful construction in homotopy theory. One reason for this is the so-called nerve theorem which provides criteria which guarantee that $N\mathcal{U}$ is homotopy-equivalent to the underlying space X (see [2]).

Example 2.6. If

$$\mathcal{U} = \{u_1 = \{1, 2, 3\},\ u_2 = \{3, 4, 5\},\ u_3 = \{4, 5, 6\},\ u_4 = \{5, 6, 7\}\},$$

then

$$N\mathcal{U} = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_2, u_4\}, \{u_3, u_4\}, \{u_2, u_3, u_4\}\}.$$

As with Example 2.3, we can represent $N\mathcal{U}$ as a graph. We leave this to the reader.

3. The Mapper Clustering Algorithm

Suppose that we are given a finite sample $S \subset X$ of an unknown population space X and a metric $d(x, y)$ evaluated only for $x, y \in X$. We now describe the Mapper clustering algorithm. First, choose a continuous map $f : X \to Z$ (evaluated only for $x \in X$), where Z is some known parameter space (usually $\mathbb{R}$). The function f is known as a *filter function*. Next, choose an open cover $V = \{v_\alpha\}_{\alpha \in A}$ of Z for some finite index set A. Since f is continuous, the sets $u_\alpha = f^{-1}(v_\alpha)$ are open in X. In general, each u_α is a union of many connected components

$$u_\alpha = u_{\alpha,1} \cup u_{\alpha,2} \cup \cdots \cup u_{\alpha,n_\alpha}.$$

Hence, we have an open cover

$$\mathcal{U} = \{u_{\alpha,i}\}_{\substack{\alpha\in A,\\ 1\leq i\leq n_\alpha}}$$

of X. The nerve $N\mathcal{U}$ is our "approximation" for X.

Since we do not know the underlying space X, we cannot construct the sets $u_{\alpha,i}$. We approximate each of these sets by

$$f^{-1}(v_\alpha) \cap S = S_{\alpha,1} \cup S_{\alpha,2} \cup \cdots \cup S_{\alpha,n_\alpha},$$

where the connected component $u_{\alpha,i}$ is replaced by a cluster $S_{\alpha,i}$ obtained by applying any clustering algorithm to $f^{-1}(v_\alpha) \cap S$. We illustrate this with a simple example.

Example 3.1. Let $X = \mathbb{S}^1$ be the 1-sphere centred at the origin, and let the parameter space be $Z = [-1, 1]$. We define the filter function $f : \mathbb{S}^1 \to [-1, 1]$ to be the projection of each point of $\mathbb{S}^1$ onto its y-coordinate. We choose an open cover V of the parameter space consisting of the intervals $v_1 = (0.33, 1]$, $v_2 = [-1, -0.33)$ and $v_3 = (-0.5, 0.5)$. This open cover induces a open cover $\mathcal{U}$ of $\mathbb{S}^1$ obtained from the sets $u_1 = f^{-1}(v_1)$, $u_2 = f^{-1}(v_2)$ and $u_3 = f^{-1}(v_3) = u_{3,1} \cup u_{3,2}$ (a union of connected components). Figure 2 displays the space $\mathbb{S}^1$, the open cover $\mathcal{U}$ and the nerve $N\mathcal{U}$.

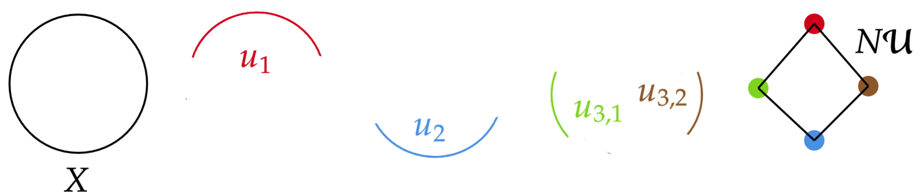


FIGURE 2. The space X, the open cover $\mathcal{U} = \{u_1, u_2, u_{3,1}, u_{3,2}\}$ and the nerve $N\mathcal{U}$ from Example 3.1.

Note that the spaces X and $N\mathcal{U}$ have the same Euler characteristic $\chi(N\mathcal{U}) = 0$. This is a consequence of the nerve theorem remarked upon in Remark 2.5: by the nerve theorem, X and $N\mathcal{U}$ are homotopy-equivalent and hence have the same Euler characteristic.

4. Application of the Mapper Clustering Algorithm to a Real-World Dataset

In this section, we apply the Mapper clustering algorithm to the Miller-Reaven diabetes study [3] using the TDAmapper package [4] for **R**. This dataset is available in **R** as `chemdiab`.

In [3], Miller and Reaven examined the relationship between blood chemistry measures of glucose tolerance and insulin in 145 non-obese adults. For each patient, six quantities were measured: age; relative weight; fasting plasma glucose level; plasma glucose response; plasma insulin response and steady state plasma glucose (SSPG) response. By analysing the two-dimensional scatterplots of all combinations of variables, Miller and Reaven were able to eliminate the first three variables thus reducing this six-dimensional dataset to a three-dimensional one. Figure 3 displays the resulting dataset.

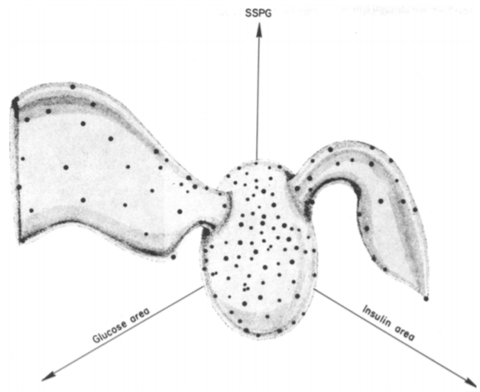


FIGURE 3. Graphical representation of the dataset for the Miller-Reaven diabetes study. This illustrates the relationship between plasma glucose response, plasma insulin response and SSPG response.

Miller and Reaven discovered the pattern shown in Figure 3, which shows a central core together with two "flares" emanating from it. The patients in each of these flares were regarded as suffering from essentially different diseases, corresponding to the division of diabetes into the adult onset and juvenile onset forms. The Mapper clustering algorithm provides us with a simple tool for detecting these "flares". Figure 4 displays the TDAmapper output for the dataset. We denote by n, p and b the number of intervals, the percentage overlap of the intervals and the number of bins when clustering, respectively, and we attach an integer label to each vertex.

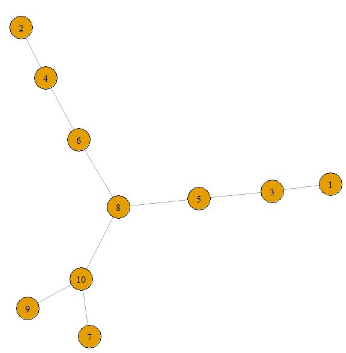


FIGURE 4. TDAmapper output for the Miller-Reaven diabetes study with $n = 5$, $p = 40$ and $b = 15$.

The TDAmapper output displayed in Figure 4 distinguishes between the two "flares" displayed in Figure 3: the edges 2-4-6-8 and 1-3-5-8 each represent a distinct "flare".

References

1. G. Singh, F. Memoli and G. Carlsson, *Topological Methods for the Analysis of High-Dimensional Datasets and 3D Object Recognition*, Point Based Graphics 2007, Prague, September 2007.
2. A. Björner, *Topological Methods*, appears in Handbook of Combinatorics, Vols. 1, 2, page 1850, Elsevier, Amsterdam, 1995, MR1373690 (96m:52012).
3. G. Reaven and R. Miller, *An Attempt to Define the Nature of Chemical Diabetes Using a Multidimensional Analysis*, Diabetologia, 16, 17-24 (1979).
4. P. Pearson, D. Muellner and G. Singh, *Package TDAmapper*, `https://cran.r-project.org/web/packages/TDAmapper/TDAmapper.pdf`.

Department of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
Email address: `g.small1@nuigalway.ie`

Department of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
Email address: `W.Leacy2@nuigalway.ie`

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0894-0347(XX)0000-0

TYCHONOFF'S THEOREM

MATTHEW VAN DER WALT AND GRACE KEELY

1. Introduction

In his 1930 paper Andrey Nikolayevich Tikhonov became the first person to prove what is now known in topology as Tychonoff's Theorem. His objective in his paper was to prove that any product of compact spaces is itself compact, Tychonoff's initial proof regarded only the unit interval but was later expanded beyond this special case. The importance of this theorem in the field of topology has been stated numerous times with Steven Willard calling it "the single most important result in general topology".

2. Definitions

In order to prove this theorem we must first critically evaluate our definitions for compactness and the product topology.

2.1. **Compactness.** The evolution of our definitions for compactness of topological spaces began in 1895 when Emile Borel stated and proved an early form of what we know today as the Heine-Borel Theorem.

Theorem 2.1 (Heine-Borel Theorem)**.** *For any subset S of Euclidean space, the two statements below are considered to be equivalent:*
S is closed and bounded.
S is compact, or, every open cover of S has a finite subcover.

With this definition we will consider a formulation of compactness in order to prove Tychonoff's Theorem:
Any topological space X is compact if and only if for each collection of open sets with the property that no finite subcollection covers there is a point $x \in X$ so that x is not covered by the collection of open sets

Definition 2.2. Let E be a subset of a topological space. We say that a limit point x of E is a perfect limit point of E if for every neighbourhood U of x the cardinality of $U \cap E$ is the same as the cardinality of E.
A topological space X is compact if and only if each infinite subset E of X has a perfect limit point.

Proof of Definition 1.2. In order to prove the definition above:
Let X be compact and let E be an infinite set with no perfect limit point.

Now, for each $x \in X$ choose a neighbourhood U_x, with the cardinality of $U \cap E$ being

Received by the editors 01/05/2021.
2010 *Mathematics Subject Classification.* Primary 54D30; Secondary 54B10 .

less that the cardinality of E. We have that the finite subcollection $U_{x_1}, U_{x_2}, ..., U_{x_n}$ is a cover of X. Thus, E is the finite union of $U_{x_i} \cap E$. Here we find a contradiction as the finite union of sets of cardinality less than E must also have cardinality of less than E.

Next we will suppose that every infinite set must have a perfect limit point. When X is not compact there will be an infinite collection $\{U_a | a \in J\}$ of open sets of X which will cover X and thus no finite subcollection covers. Moreover, we can now also assume that our set J has the minimum cardinality with this property.

We will now suppose that J is a well ordered set such that for each a the cardinality of $\{U_b | b < a\}$ will be less than the cardinality of J with $U_a \not\subset \cup\{U_b | b < a\}$.

Finally, we shall define the set $E = \{x_a | a \in J\}$ which gives us $x_a \in U_a \cup \{U_b | b < a\}$. Now we know that the cardinality of E and J are the same. If our point x is in X, then x will lie in some U_a, however at the start of our proof we had that the cardinality of $U_a \cap E$ is always less than the cardinality of E. This contradicts that every infinite set has a perfect limit point. □

2.2. **Product Topology.** Our topological definition of the product space is the Cartesian product of a family of topological spaces equipped with the product topology.

Definition 2.3 (The Product Space)**.** Let $X_1, X_2, ..., X_n$ be topological spaces. A subset U of the Cartesian product $X_1 \times X_2 \times ... \times X_n$ is said to be *open* (with respect to the product topology) if, given any point p of U, there exist open sets V_i in X_i for $i = 1, 2, ..., n$ such that $\{p\} \subset V_1 \times V_2 \times ... \times V_n$.

3. Tychonoff's Proof

Theorem 3.1. *Let X and Y be compact spaces then $X \times Y$ is compact*

Proof. Let E be an infinite subset of $X \times Y$.

Firstly, we will show that there exists an $a \in X$ so that for each neighbourhood U of a we have that the cardinality of $(U \times Y) \in E$ has the same cardinality as E itself. If we find that no such $a \in X$ exists, then we know that for each $x \in X$ we must have some open set U_x containing x such that our set $(U_x \times Y) \cap E$ will have a cardinality smaller than E. By using our above formualtion of compactness the finite subcollection $U_1, U_2, ..., U_m$ covers X.

Hence, $E = (X \times Y) \cap E = ((U_{x_1} \cup U_{x_2} \cup ... \cup U_{x_m}) \cap E) = \bigcup_{i=1}^{m}((U_{x_i} \times Y) \cap E)$. This is actually a contradiction because the infinite set E obviously cannot be written as a finite union of sets.

Finally, we can show that there exists $a, b \in Y$ so that, for each open set of the form $U \times Y$ containing $(a, b), (U, V) \cap E$ has the same cardinality as E. This now implies that (a, b) is a perfect limit point of E, so we have that $X \times Y$ is compact. □

4. A simpler proof of Tychonoff's theorem

With Tychonoff's own theorem stated and proved above I will provide a more simple and hopefully easier to understand proof of this same theorem.

Theorem 4.1. *Let X and Y be compact spaces then $X \times Y$ is compact.*

Proof. Firstly, we will consider that the set $X \times Y$ cannot be covered by a finite union of open subsets $S \in X \times Y$.

Similar to the proof above, we know that there is no neighbourhood U in X where S can be found to be a cover of $U \times Y$.

Next we show that there is no open set $U \times V$ which contains an element $a \in X$ and $b \in Y$ that is covered by our union of open subsets S.

Leading us to the inevitable conclusion that, our point $(a.b)$ cannot be covered by any finite union of open sets S, meaning that our product topology $X \times Y$ is in fact, compact. □

5. Applications

Tychonoff's theorem has been used to prove numerous other mathematical theorems. These include theorems about compactness of certain spaces, compactness of the unit ball of the dual space of a normed vector space and the Arzelà–Ascoli theorem which characterizes the sequences of functions in which every subsequence has a uniformly convergent sequence.

Tychonoff's theorem has also been used to prove other theorems less evidently linked with compactness, such as the De Bruijn- Erdős theorem, which states that when "k is a positive integer, and with a graph G having the property that any finite subgraph of G is k-colourable. Then G is k-colourable itself." This theorem relates graph colouring of an infinite graph to the same problem on its finite subgraphs. It states that, when all finite subgraphs can be coloured with k colours, the same is true for the whole graph.

A sort of unwritten rule is that Tychonoff is likely to be used for any sort of construction that takes a fairly general object as its input and outputs a compact space. An example of this is the Stone space of maximal ideals of a Boolean algebra.

References

1. P. Dugac, *"Sur la correspondance de Borel et le théorème de Dirichlet–Heine–Weierstrass–Borel–Schoenflies–Lebesgue"*, Arch. Int. Hist. Sci, 1989.
2. David G. Wright, *Tychonoff's Theorem*, 1991.
3. Stephen Willard, *General Topology*, 2004. Mineola, N.Y.: Dover Publications. ISBN 978-0-486-43479-7. OCLC 115240.
4. A. Tychonoff, *Uber die topologische Erweiterung von Raumen*, Math. Ann. 102 (1929), 544-561.
5. N. De Bruijn and P. Erdos *colour problem for infinite graphs and a problem in the theory of relations*, Nederl. Akad. Wetensch. Indag. Math. 13 (1951), 371-373.

ABSTRACT. In this article we will outline the topological theorem first stated and proved by Andrey Nikolayevich Tikhonov (whose family name is sometimes spelled Tychonoff). Firstly we will show the initial 1930 proof by Tychonoff himself, followed by a simpler proof that may be easier to understand for my fellow students of topology. Finally, we have outlined some of the applications of Tychonoff's Theorem which will hopefully illustrate the importance of the theorem and provide context for its uses.

THUMNASNAGH, CLOONIFF, MOYCULLEN, CO.GALWAY
Email address: m.vanderwalt2@nuigalway.ie

BALLYCORRIGAN, BALLINA, CO.TIPPERARY
Email address: g.keely1@nuigalway.ie

An Investigation Into Topological Data Analysis To Predict Stock Market Crashes

Emma Corbett, Anna Golden, and Lydia Costello
(Communicated by Ralph Cohen)[1]

Abstract

For this project, we wanted to investigate if there was an efficient way to monitor the stock market. In doing this we found that Topological Data Analysis (TDA) could be used to identify worrying trends. This involves looking into the daily returns of major stock market indicators during a number of historical stock market crashes. We analyze and compare Gidea and Katz's [1] method and De Oliveira's [2] method. Using De Oliveira's [2] code, we also applied this knowledge to the recent Covid-19 Crash. Topological Data Analysis and persistence homology should be used alongside usual statistical measures for the most accurate econometric analysis. Using this method, monitoring the stock market is made more effective and future stock market crashes can be identified sooner.

Introduction

Billions of dollars and millions of hours are spent every year in attempts to predict the stock market. Financial analysts and statisticians do everything possible to gather data, clean it and train models to avoid substantial losses. In our history, using the power of data was considered a game-changer. Nevertheless, data collection is still a huge problem. The prediction models scientists have been working with rely on a lot on data, and the more complicated the model, the more data needed. Heavy loads of data can be challenging to gather and clean. Cleaning data is also one significant second step before modeling the p-value. If we were working with collected data from nature, it would most likely end up with impure data-based full of outliers and noise. The question is: Can there be a model that can endure the impure and generate a model with reasonable accuracy? We will highlight the importance of the TDA method compared to the traditional methods. Firstly, we will explain how each one works, especially how the baseline and the TDA-pipeline work. Secondly, the results. Finally, we will conclude with illustrations showing the strengths, weaknesses and potential of TDA in predicting stock crashes.

Methods

Now let us see more methods and how they work. However, for more simplification, we will focus only on the economic and financial side.

[1]2020 Mathematics Subject Classification. Primary 54H08.

Normal methods

- Random matrix theory (RMT): It is a theory combining nuclear physics and statistical mechanics applied to the stock market. The concept of this method is treating noise as a kind of symmetry; this allows them to discriminate between noise and signal.

- Planar maximally filtered graph (PMFG): It is an information filtering approach that projects important correlations onto a sphere to offer a better understanding for the correlated price movements between stocks.[8]

- Simple baseline: This approach does not use complicated mathematical tools; it only tracks the first derivative of average price values of stocks.

TDA Method:

TDA is a mathematical apparatus developed by Herbert Edelsbrunner, Afra Zomorodian, Gunnar Carlsson, and his graduated student Gurjeet Singh. It was popularized by Carlson's paper that turned TDA into a hot field in applied mathematics, with many applications in data analytics. The foundations of TDA had been laid years before by others in the fields of topology, group theory, linear algebra, and graph theory.

So far, the best candidate model is the Giotto-TDA, which is a high-performance topological machine learning toolbox in Python [10]. TDA is a relatively new approach that uses results of topology to treat the data collected. TDA lets the data speak for itself on deciding which topological space to use for the projection. The noise does not bother the information in this method. There are low chances for us to get a false warning due to wrong information, this is thanks to the solid mathematical tools used as for persistent homology. The massive data does not present a problem, each matrix or set can be related to topological spaces with high dimensions, and it is easy to manipulate

TDA Applied to Financial Time Series Data

In this section we discuss Gidea and Katz's [1] paper and their proposed application of TDA, specifically persistent homology, to financial time series data in order to predict stock market crashes. Gidea and Katz [1] analyzed the 2000 Dot-Com Crash and the 2008 Financial Crash.

Initial Data and Objectives

Gidea and Katz [1] applied TDA to a multidimensional financial time series, in order to determine whether it could be used to detect an "increasing systemic risk in financial markets". They analysed a $4D$-point cloud consisting of daily log-returns of the S&P 500, DIJA, NASDAQ and Russell 2000, the four primary US stock market indices. They used a sliding window of $\omega = 50$ trading-days, with sliding step of one day, to obtain a $4D$-point cloud for each instance of the window. Studies and analysis of stock markets have found that early warning signals for a financial crash include: a period of increased variance in stock market indices and a shifting of spectral density of time series towards low frequencies. Spectral density is a statistical technique whereby time series data is transformed from time domain to frequency domain, in order to make observations about its periodicity [3]. Gidea and Katz [1] apply TDA to detect these early warning signals of a crash.

Simplicial Complexes

A simplicial complex is an abstract collection of entities, which consists of nodes $(i, j, k, \cdots)$ or sets of nodes $(\{i\}, \{i, j\}, \{i, j, k\}, \cdots)$ [9]. These collections can be used to construct links, surfaces, and higher-dimensional objects. For example, we can decompose an arbitrary simplicial complex into its $0-$simplexes (*nodes*), $1-$simplexes (*links*), $2-$simplexes (*faces*), $3-$simplexes (*tetrahedrons*) components. In other words, simplexes are generalizations of a triangle in arbitrary dimensions, and a simplicial complex is an outcome of performing triangulation in arbitrary dimensions of the raw data [9]. The simplicial complex is a unique signature that characterizes the topological structure of the data.

Vietoris-Rips Simplicial Complex:

We have a point cloud data set, $X = \{x_1, x_2, ..., x_n\}$ in $\mathbb{E}^d$. In this case $n = 4$. We associate a topological space to X by defining a Vietoris-Rips simplicial complex, $R(X, \epsilon)$ for each distance $\epsilon > 0$ as follows: "for each $k = 0, 1, 2, ...$, a k-simplex of vertices $\{x_{i1}, ..., x_{ik}\}$ is part of $R(X, \epsilon)$ if and only if the mutual distance between any pair of its vertices is less than ϵ, that is

$$d(x_{ij}, x_{il}) < \epsilon, \text{for all } x_{ij}, x_{il} \in \{x_{i1}, ..., x_{ik}\} \text{ "[1]}$$

In simple terms, imagine ϵ defines the radius of an imaginary ball centered at each of the points in the cloud. As ϵ increases, the ball will grow outwards and touch other balls.

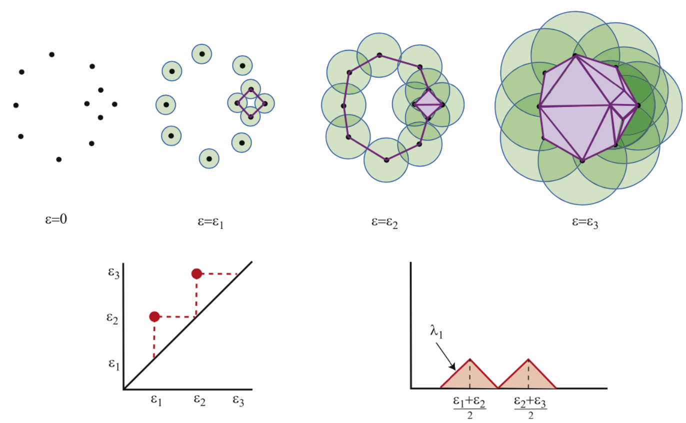


Figure 1: Example Vietoris-Rips Simplicial Complex for different values of ϵ with corresponding (L) persistence diagram and (R) persistence diagram. [1]

Persistent Homology

We now use this topology to compute persistent homology. "Homology associates a number of algebraic objects to topological spaces. [It] is a mathematical way of counting different types of loops and holes in a topological space"[4]. Homology is a topological invariant [4]. Persistent homology analyzes the persistence of k-dimensional holes such as connected components ($k = 0$) and loops ($k = 1$) in a topological space. Persistence is calculated using a resolution parameter, ϵ. As ϵ varies different k-dimensional holes appear and then disappear under the Vietoris-Rips simplicial complex [1]. The value of ϵ when the hole appears is the 'birth' value, b_α, and the value when it disappears is the 'death' value, d_α . The range of these values is a measure of persistence. Holes that persist for a large range of resolutions are more significant [1].

Persistence Diagrams and Persistence Landscapes:

Persistence diagrams and persistence landscapes are two methods used to analyse persistence. Persistence diagrams plot the birth value vs. the death death value and have a natural metric space. Persistence landscapes consist of a sequence of continuous, piecewise linear functions that transform the birth-death coordinates and are embedded in a Banach space. The piecewise, linear function used by Gidea and Katz [1] is defined as:

$$f_{(b_a,d_a)} = \begin{cases} x - b_\alpha & \text{if } x \in (b_\alpha, \frac{b_\alpha+d_\alpha}{2}] \\ -x + d_\alpha & \text{if } x \in (\frac{b_\alpha+d_\alpha}{2}, d_\alpha) \\ 0 & \text{if } x \notin (b_\alpha, d_\alpha) \end{cases}$$

The Banach space (a complete, normed vector space) used in this method is the L^p-space, which has the p-norm. The p-norm of a vector $x = (x_1, ..., x_n)$ is defined as:

$$||x||_p := (\sum_{i=1}^{n} |x_i|^p)^{1/p}$$

The L^p-norm for p=1 is the taxicab norm and for p=2 is the Euclidean norm [5]. Both persistence diagrams and landscapes are robust under perturbations of the underlying data, ie. they shift only slightly if the underlying data changes slightly [1]. Gidea and Katz [1] applied this method to their data and obtained the following persistence diagrams and landscapes:

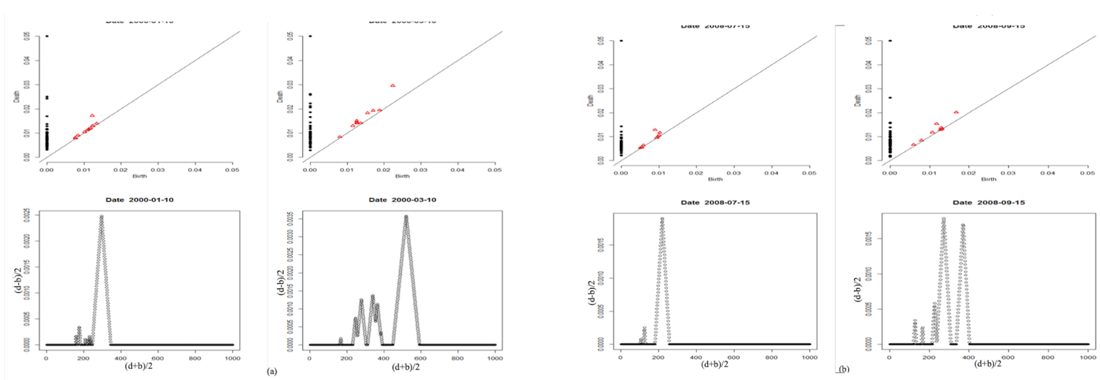


Figure 2: Rips persistence diagrams and the corresponding persistence landscapes. The black dots represent connected components and the red triangles represent loops. (a) Dot-Com Crash 2008, (b) Financial Crash 2000. [1]

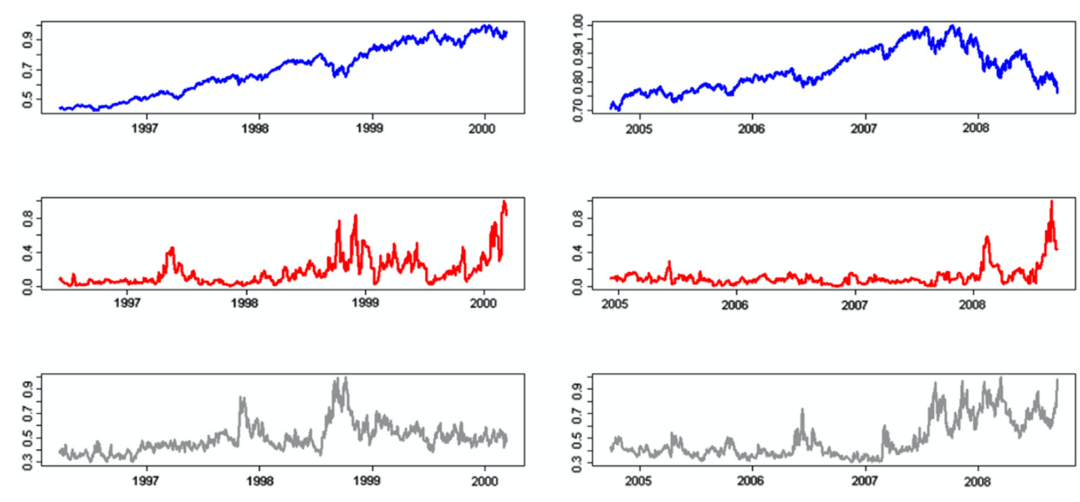


Figure 3: 1000 trading-days prior to (L) the Dot-Com Crash, 2000 and (R) Financial Crash, 2008. (Blue) Normalised daily SP 500 returns; (Red) Normalised L^1-norm of the topological landscapes above; (Grey) VIX, a measure of expected market volatility. [1]

They [1] then computed the L^1-norm of the persistence landscapes in Figure 2 for the 1000 trading days prior to either crash and created a new time series of the L^1-norm data. We can see some indications of growth of the normalised L^1-norms prior to both crashes (Figure 3).

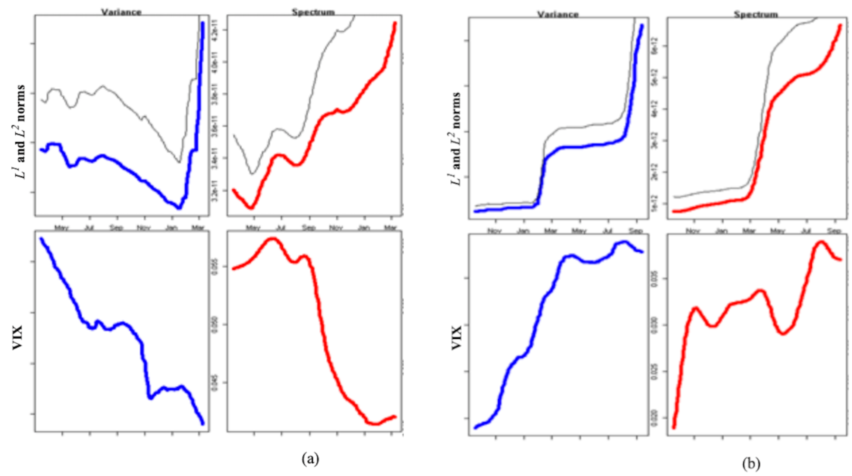


Figure 4: 250 trading-days prior to (a) 10/03/2000 the Dot-Com crash and (b) 15/08/2008 the Lehman Brothers bankruptcy. (Top) L^1 and L^2-norm. (Bottom) VIX. (Blue) variance and (Red) average spectral density of the time series of L^1 and L^2-norms of the above persistence landscapes. [1]

Finally, Gidea and Katz [1] computed the variance and average spectral density of the L^1 and L^2-norms. They found that the average spectral density of the L^p-norms exhibit strong growth prior to the stock market peak. They also found that this trend can be seen up to 250 days before either the 2000 or 2008 crashes (Figure 4).

De Oliveira [2] Approach and Findings:

De Oliveira [2] based his method on Gidea and Katz's [1] method and shared his code on Github. This allowed us to examine if his model predicted the 2020 Covid-19 Crash. He compared his model to a baseline model of the first-derivative of the S&P 500 index. De Oliveira's [2] model is a less sophisticated version of Gidea and Katz's [1].

Baseline method:

The baseline method is achieved by taking the first derivative of the S&P 500 and normalizing it to take values in the [0,1] interval. De Oliveira [2] then set a threshold of 0.3. For any movement above this line the baseline method predicts a crash. Even this naive and straightforward method could detect the 2000 and the 2008 crashes. However we can see it is noisy, overly sensitive and inaccurate: the method indicated an interval of years that could witness crises. The problem with noises is that they announce false alarms.

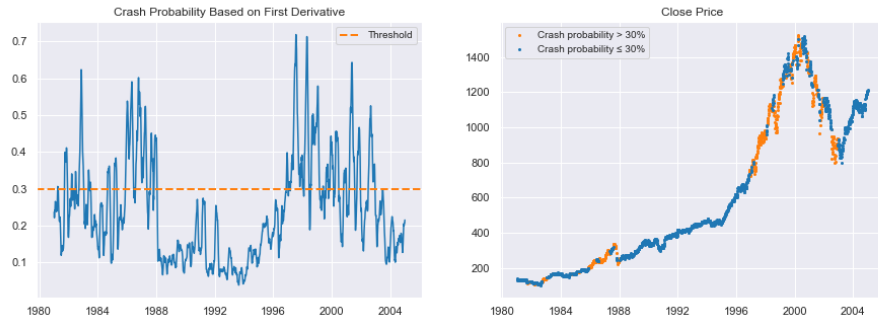


Figure 5: Baseline Method

Taken's Embedding

De Oliveira's [2] first step was to generate a point cloud from the S&P 500 index time series data. He used Taken's embedding to do this. A different method than that used by Gidea and Katz, who chose to represent their data in a low dimensional space, $\mathbb{E}^d$, with $d << \omega$, $d = 4$, $\omega = 50$. Time series are a sequence of data points in successive order over a period of time. Tracking those series is easy locally because most of the time they are globally represented as scatter plots in two dimensions. The point is to capture the periodic behavior of the signals. However, unfortunately, the data is not always represented in two dimensional sets, and this is where we need to represent a univariate time series as a point cloud i.e. a set of vectors in a Euclidean space of arbitrary dimension.

We use this procedure: let d and τ be two integers. for each time $t_i \in \{t_0, t_1, \cdots\}$, we can collect the values of the variable y at d distinct times, evenly spaced by τ and starting at t_i to represent them as a vector in $d-$ dimensional space:

$$Y_{t_i} = \Big(y_{t_i}, y_{t_i} + \tau, \cdots, y_{t_i} + (d-1)\tau\Big)$$

τ : *the time delay parameter* $\qquad$ d : *the embedding dimension*

After we find the vectors, we apply Takens's theorem. This step is called Time-delay embedding or Taken's embedding. This process is applied separately on a series of sliding windows of length ω time units and a sliding step of one time unit. This leads to a time series of point clouds of data from ω trading days.

TDA Method

After using Taken's embedding to create a times series of point clouds, De Oliveira [2] then applied persitent homoglogy, created the persistence landscapes and calculated the landscape distances (L^1-norms). De Oliveira's [2] landscape distance graphs as seen in Figure 6 and 7, are almost the same as the normalised L^1-norm of the topological landscapes in Figure 3. Instead of computing the variance and average spectral density of this data, which was the predictor Gidea and Katz [1] found most accurate, De Oliveira [2] sets a threshold of 0.3 above which he predicts a crash is imminent. In the following diagrams we will see the results of this method. It accurately predicts the 2000 Dot-Com Crash as seen in Figure 6. It also predicts the 2008

Financial Crash but too late, the market is almost at its lowest point before the threshold is crossed. This method gives better precision than the baseline method and significantly less noise but is less effective than Gidea and Katz's method.

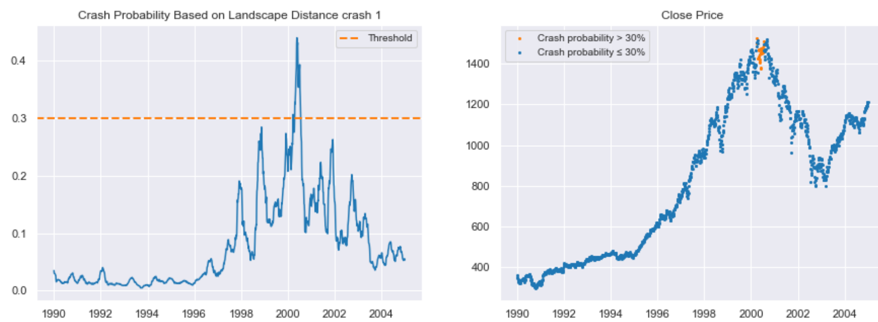


Figure 6: De Oliveira TDA Method: Dot-Com Crash, 2000 [2]

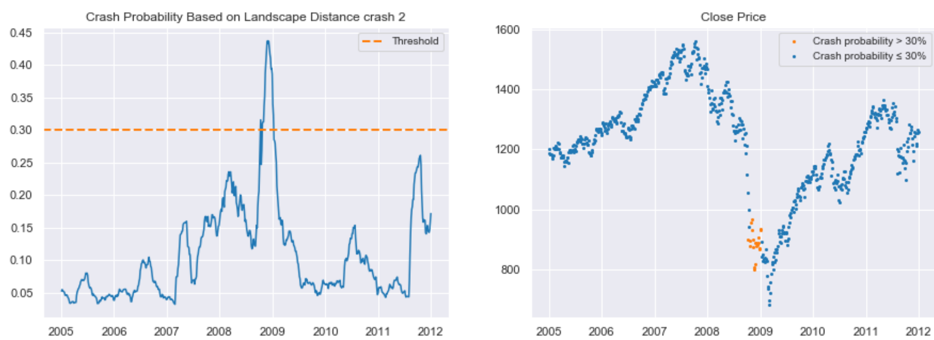


Figure 7: De Oliveira TDA Method: Financial Crash, 2008 [2]

Applying De Oliveira's [2] method to the Covid-19 Crash

We adapted the code shared by De Oliveira [2] to apply to the Covid-19 Crash. The results can be seen in Figure 8. The method clearly did not predict the Covid-19 crash. It is possible that this is because the method was designed to detect 'an increasing systemic risk in financial markets' [1] and the Covid-19 Crash was due to a health crisis, rather than a financial asset price bubble as in 2000 or 2008.

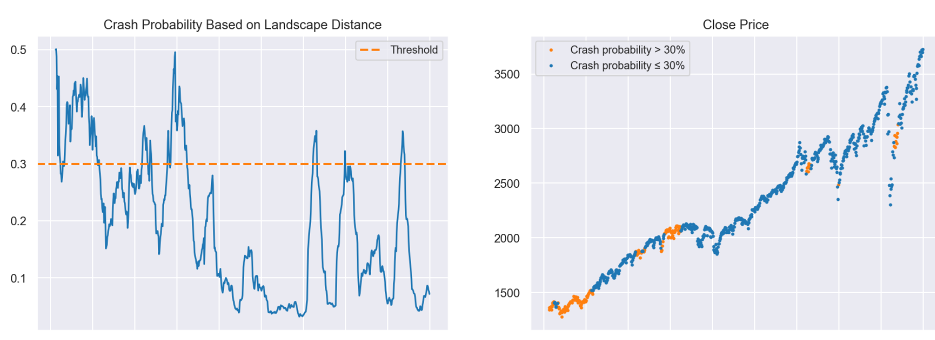


Figure 8: Covid-19 Crash, 2020

Conclusion

While it is next to impossible to predict a stock market crash, and anyone who could do so would most likely choose to make billions rather than share their findings publicly, we found that Gidea and Katz's [1] method of computing the average spectral density of the L^p proved an effective early warning signal for the 2000 and 2008 crashes. We compared this to De Oliveira's [2] method, which was less efficient. We applied De Oliveira's [2] method to data from 2020 and found it was unsuccessful at predicting the Covid-19 Crash. An interesting next step would be to compare both methods [1, 2] applied to Covid-19 data.

Gidea and Katz present a novel way of applying TDA to time series data that has applications in finance as well as other fields. For example, a more recent paper by Gidea [6] applies similar methods to predicting cryptocurrency crashes. Another paper by Guo [7], builds on their method and applies TDA to study the interdependence of financial markets during financial crises, particularly the Covid-19 Crash. Overall, we found that the TDA method is robust to perturbations and efficient at removing noise. Gidea and Katz's [1] method has potential to detect early warning signals in the financial sector and complement existing methods.

References

[1] `Gide, M., Katz, Y.` (2018). Topological data analysis of financial time series: Landscapes of crashes. *Physica A*, 491 820-834.

[2] `De Oliveira, W.L.` (2019) Detecting stock market crashes with topological data analysis *Towards Data Science* Available at: `https://towardsdatascience.com/detecting-stock-market-crashes-with-topological-data-analysis-7d5dd98abe42`

[3] `Jones, J.H.` (2018) Time Series and Spectral Analysis *Stanford University* Available at: `http://web.stanford.edu/class/earthsys214/notes/series.html`

[4] `Stolz, B. Mahler, B.` (2021) H is for Homology *Mathematical Institute, University of Oxford* Available at: `https://www.maths.ox.ac.uk/about-us/life-oxford-mathematics/oxford-mathematics-alphabet/h-homology`

[5] `Wikipedia` (2021) L^p space. Available at: `https://en.wikipedia.org/wiki/Lp_space`

[6] `Gidea, et al.` (2020) Topological recognition of critical transitions in time series of cryptocurrencies *Physica A* 548

[7] `Guo, et al.` (2021) Analysis of global stock markets' connections with emphasis on the impact of COVID-19 *Physica A* 569

[8] `N. Vandewall, F.Brisbois, X.Tordoir` (2001) *Rechearch paper: Non-random topology of stock markets*

[9] https://www.sciencedirect.com/topics/computer-science/simplicial-complex

[10] giotto-tda: : A Topological Data Analysis Toolkit for Machine Learning and Data Exploration :Guillaume Tauzin; 22(39):16, 2021.https://www.jmlr.org/papers/volume22/20-325/20-325.pdf

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

APPLICATIONS OF THE BORSUK-ULAM THEOREM

JOSHUA CONNEELY AND RACHEL KELLY

(Communicated by Julie Bergner)

Abstract. The Borsuk-Ulam Theorem is used in many proofs and in many areas of mathematics. In this paper we will discuss The Borsuk Ulam Theorem and the different variations of the theorem, and the many mathematical applications of it. This includes some topological examples as well as how the theorem can be used to prove many other theorems.

1. History

The Borsuk-Ulam Theorem was first conjectured by Stanislaw Ulam and was then proven by Karol Borsuk in 1933. It is a very useful theorem with many different applications, including combinatorics, differential equations and even economics.
There are many equivalent but different formulations of this theorem. We will mainly discuss only one in this project.
There are also several proofs of the Borsuk–Ulam theorem, however we will not prove this here.

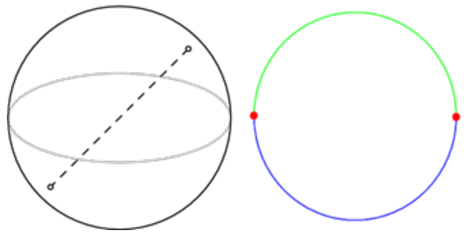
Karol Borsuk(right) and Stanislaw Ulam(left)

2. The Theorem

There are many equivalent forms of the Borsuk Ulam Theorem, six of which are listed below. Before we can understand the theorem, we need a definition. There are various proofs explaining how each of the following theorems are equal, however we will not prove this and instead focus on understanding and applying Theorem 2.2 below.

Received by the editors April 29, 2021 .
2000 *Mathematics Subject Classification.* Primary 54H25, 55M200; Secondary 47H10.

Definition 2.1. In mathematics, antipodal points of a sphere are those diametrically opposite to each other (the specific qualities of such a definition are that a line drawn from the one to the other passes through the centre of the sphere so forms a true diameter). This term applies to opposite points on a circle or any n-sphere. Below there is examples of antipodal points on a circle and a sphere. [1]



Theorem 2.2. *For every continuous mapping $f : S^n \to R^n$ there exists a point $x \in S^n$ such that f(x) = f(-x).*

Theorem 2.3. *For every antipode-preserving map $f : S^n \to R^n$ there is a point $x \in S^n$ satisfying f(x) = 0.*

Theorem 2.4. *There is no antipode-preserving map $f : S^n \to S^n - 1$.*

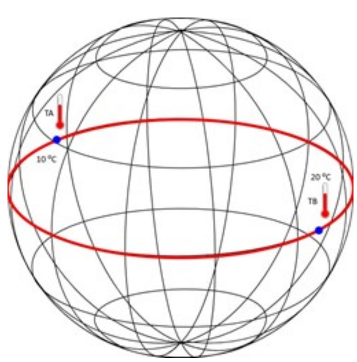
Theorem 2.5. *There is no continuous mapping $f : B^n \to S^n - 1$ that is antipode-preserving on the boundary.*

Theorem 2.6. *A covering of S^n by n + 1 closed sets, F_1, . . . , F_{n+1}, has at least one set that contains a pair of antipodal points.*

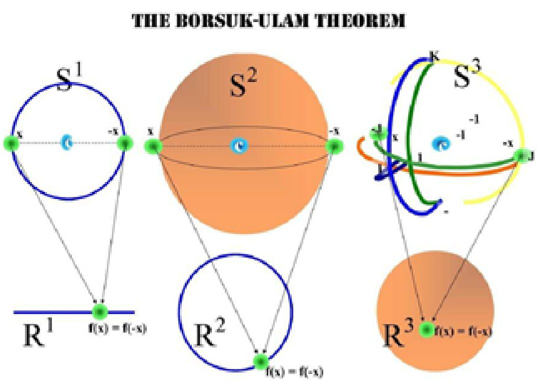
Theorem 2.7. *A covering of S^n by n + 1 open sets, U_1, . . . , U_{n+1}, has at least one set that contains a pair of antipodal points.*

3. An Informal Description

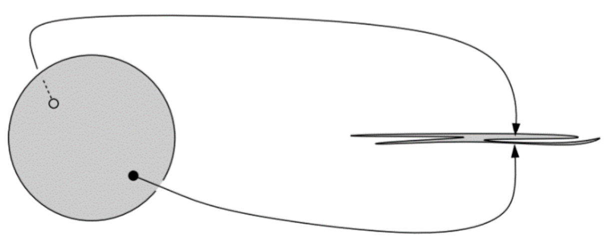
Example 3.1. For n=1, a nice example to illustrate this theorem is that on the equator there must exist opposite points with the same temperature. We can visualise how this is true: if we take a random pair of opposite points on the equator, A and B, and suppose A starts out warmer than B. As we move A and B together around the equator, you will move A into B's original position, and simultaneously B into A's original position. But by that point A must be cooler than B. So somewhere in between they must have been the same temperature!



Example 3.2. For n = 2: Take a sphere, deflate and crumple it, and lay it flat: Then there are some two points on the surface of the ball that were antipodal, and are now lying on top of one another on the same point! This is easy to visualise on the earth, imagine the earth as a smooth ball with centre on the origin. Then, for example, take our points as the north and south pole. If we simply compress the sphere down from top to bottom into a flat circle, it is clear to see that the north and south poles are now one point, in this circle.
In the below image there is a example of antipodal points on the circle when it is deformed into a line, an example of the sphere when it is deformed into a circle, and also an example of s^3 being deformed into a sphere. [2]



While this seems obvious for these chosen points, the Borsuk-Ulam theorem proves it is actually true that some two antipodal points will always converge to one point no matter what continuous deformation you take of the sphere. [3]
Even in this example below, it is a strange way of deforming the sphere but since it is continuous and goes from S^2 to R^1 there is two antipodal points in the sphere that are mapped to the same point on R^1.



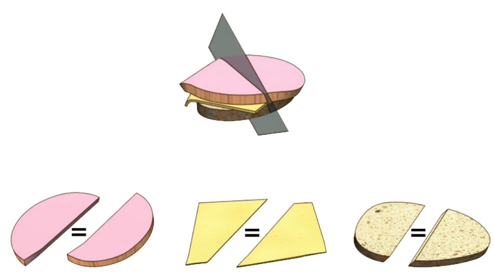
Example 3.3. Yet another popular interpretation for n=2 involves the Earth. It says that at any given time there are two antipodal places on the sphere that have the same temperature and, at the same time, identical air pressure.[2]

4. Mathematical Applications of This Theorem

Example 4.1 (Example The Brouwer Fixed Point Theorem)**.** Recall every continuous mapping f: $B^n \to B^n$ has a fixed point: f(x) = x for some $\in B^n$. An informal description of the Brouwer Fixed Point Theorem for n=3 is as follows: no matter how we move around a liquid in a glass (as long as we do it continuously), some point is always in the same position it was before the moving took place (although it might have moved around in the meantime). Moreover, if we try to move this point out of its original position, we will unavoidably move some other point back into its original position. This theorem looks quite similar to the Borsuk-Ulam Theorem, and in fact it is: the Borsuk-Ulam Theorem can give us a proof of the Brouwer Fixed Point Theorem. The Brouwer Fixed Point Theorem is very useful in topology and other areas of mathematics, so we see the Borsuk Ulam Theorem is useful as it can be used to prove it. [3]

Definition 4.2. A hyperplane in R^d is a (d-1)-dimensional affine subspace, or equivalently, the set $\{x \in R^n : \langle a, x\rangle = b\}$ for some a $\in R^d$ and b $\in R$, where $\langle , \rangle$ is the standard inner product on R^n. A hyperplane defines two closed half-spaces of the form $\{x \in R^n : \langle a, x\rangle \leq b\}$ for some a $\in R^d$ and b $\in R$.[3]

Example 4.3 (Example The Ham Sandwich Theorem)**.** The following informal statement that gave the ham sandwich theorem its unusual name: For every sandwich made of ham, cheese, and bread, there is a planar cut that simultaneously halves the ham, the cheese, and the bread.



The mathematical ham sandwich theorem says that any d (finite) mass distributions in Rd can be simultaneously bisected by a hyperplane: This geometric result has many interesting consequences and its proof is given by using the Borsuk-Ulam theorem.
The ham sandwich theorem can be proved for the n-dimensional variant using the Borsuk–Ulam theorem. This proof follows the one described by Steinhaus and others (1938), attributed there to Stefan Banach, for the n = 3 case.

An interesting application of the Borsuk Ulam theorem and the Ham Sandwich Theorem Every open necklace with d kinds of stones can be divided between two thieves using at most d cuts. [3]
In the below image we have 4 different coloured stones and have demonstrated how each thief can get the same number of each jewel by cutting the necklace only four times! It is always true that no matter how many different stones you have, say d, and no matter which way they are ordered it will always take at most d cuts to divide them between two thieves evenly.

Example 4.4 (Example The Kneser Conjecture). A final important application of The Borsuk-Ulam Theorem is The Kneser Conjecture.
Consider a set with 2n+k elements, and its subsets which contains n-elements. Kneser's Conjecture states that the minimum number of classes that are required to partition these subsets so that intersecting pairs of subsets in the class is non-empty, is k+2. Any partition with fewer classes results in at least one class with a pair of disjoint subsets. How the Borsuk-Ulam Theorem proves Kneser's Conjecture is not evident, but it is true. [3]
Kneser's conjecture has many applications, mainly in graph theory and the colouring of maps. We can hence see that the Borsuk-Ulam Theorem has many applications and can be used across mathematics.

References

1. Reference (2008) The Borsuk–Ulam Theorem. In: Using the Borsuk–Ulam Theorem. Universitext. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-540-76649-0_2
2. Amer. Math. Monthly 104 (1997), 855-859. BORSUK-ULAM IMPLIES BROUWER: A DIRECT CONSTRUCTION FRANCIS EDWARD SU Relevant page in Armstrongs book: 215 9.3
3. Some applications of the Borsuk-Ulam Theorem Math REU University of Chicago Apprentice Program 2017 Jackson Dougherty September 25, 2017

Email address: j.conneely6@nuigalway.ie

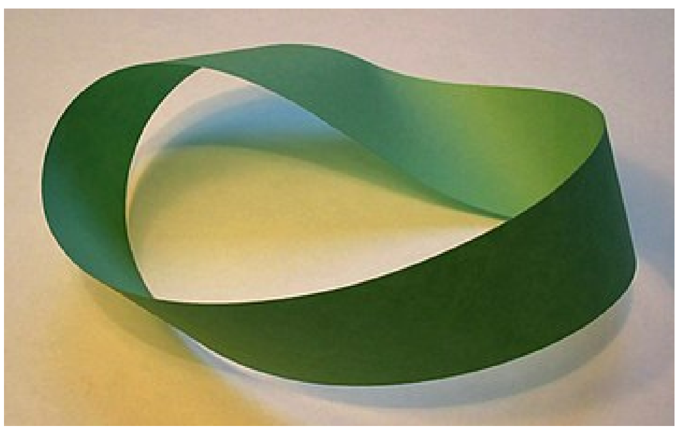
Email address: r.kelly65@nuigalway.ie

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

TOPOLOGY? WHAT IS IT AND WHAT MAKES IT SO DIFFERENT TO GEOMETRY

DYLAN CASSIDY KILLANE 18468602, CIAN FORDE 18424894,
AND LIAM LUPTON-SMITH 18341696

Abstract. From the moment we entered into school in some way we've been studying Geometry. Geometry is the study of shapes and everything about them, their sizes, shapes, positions, angles and dimension. But what is topology and what makes it so different to Geometry, is it so different to geometry? That is what our article today will delve into with the help of different sources such as the 2007 movie Flatland directed by Dano Johnson which is based off of the 1884 novel Flatland: A Romance Of Many Dimensions by Edwin Abbott Abbott.



1. What Is Topology?

According to the Oxford dictionary Topology is "the study of geometrical properties and spatial relations unaffected by the continuous change of shape or size of figures". In Topology, we deal with abnormal shapes like the Mobius Strip or the Torus. We study the properties of these shapes like you would in geometry with circles, squares or triangles. For example, the previously mentioned Mobius strip, pictured above, has one side and one edge and has been twisted to make one loop which if you were to put a car toy on and go around it would take two rotations to return to your starting point. There are a lot of similarities to Geometry in topology however the two are also in a way completely different as you will soon find out.

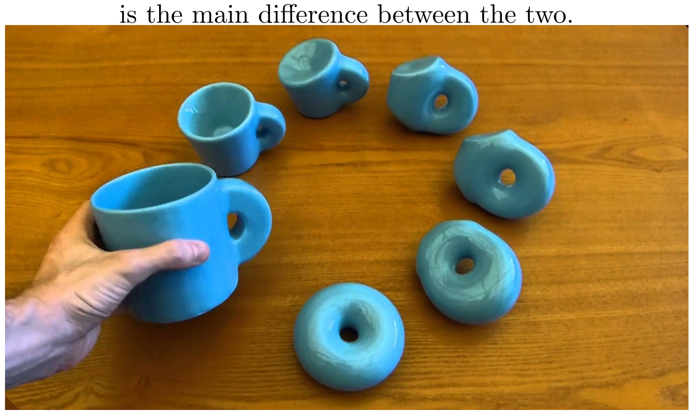
2020 *Mathematics Subject Classification.* Primary .

2. Understanding Topology - The Fishing net Example

Imagine a fishing net like the one below,When you lay the fish net out flat we see that it is just a massive collection of connections made by the rope. If you were to break one of the connections, then in all likely hood the net is beyond repair and needs to be replaced. Topology studies everything you can do to this fishing net without breaking the connections (as this defies the principles of topology). This means that you can fold it, push it around in various ways, get on multiple sides and make waves with the fishing net, and do many other things to it. All of this is deforming the fishing net from the flat sheet that it originally was, but never once does it break any of the connections that make it up.If you break the network connections, then it's likely you are either: not talking about topology anymore, or you are talking about very specific topological concepts related to very specific ways to break a network. This is also why topology is related to Knot Theory and mathematical Origami.

3. What differentiates it from Geometry?

The big way to distinguish topology from geometry is how topologists and geometrists look at their shapes and objects. For example, a topologist will look at a doughnut and a coffee mug and see two equivalent objects, this is because a coffee mug has one hole just like a doughnut and therefore we can bend and deform one of the two to look like the other (the process for this is shown in the image below). However a topologist obviously knows that these two objects aren't the same but in topological terms they are and this is why a geometrist won't look at these two objects as equivalent as in geometric terms they are not and this is the main difference between the two.



4. Indroduction to Flatlands and its displays of dimensions

Prior to this project one of our colleagues had mentioned how they had watched the movie ”Flatlands” while in secondary school and felt that it may be of use to us topologically. Flatland: A Romance of many dimensions is a Novella written by Edwin Abbott Abbott. Since it has seen movie adaptations in Flatland: The Movie (2007) and Flatland: The Film(2007). We viewed Flatland: The Film, however the the narrative is similar across all three mediums. The story follows ”A square” and his family who live on flatland, a two dimensional space in which our protagonists live. Among those who live there are circles, lines, squares , triangles and other polygons. The backdrop for the film is a government in turmoil, this however is not relevant for the mathematical aspects of the film and is more so a segway for the film to divulge on the idea of other dimensions. The film first explores what it might be like to live in such a dimension. In a lesson with his grand daughter, A square teaches her how to see and perceive depth. As they live on a flat plane, all they can see is edges opposed to the top down view that the movie gives the audience.

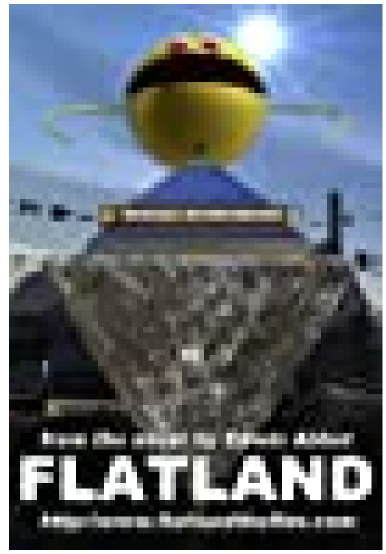
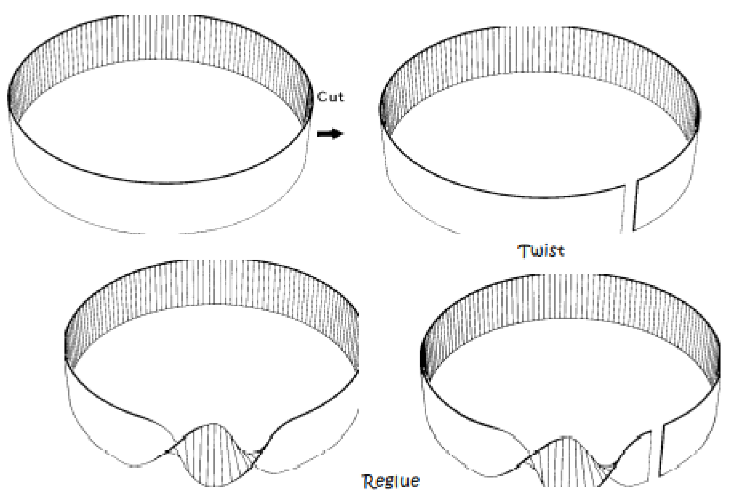


FIGURE 1. Flatland: The Film

One evening in a dream, A square visits Lineland. He meets the king of lineland who struggles to be convinced of a second dimension in which a y axis exists. From A squares perspective he can see the whole line from his point of view , and even goes through linelands x axis to prove to the king of his existence. The next time we visit another dimension is the following evening when A square is visited by a voice, A Sphere. A Sphere proclaims A square as his apostle of the three dimensions. Like A square in Lineland, A Sphere goes through Flatland and displays how big and small of a circle he can be depending on how far he goes through. He lifts A Square out of Flatland and into Spaceland, allowing A square to look at Flatland from above. Here he can see the inside of his friends and family and this allows him to find his brother who is missing in the subplot. After a ruckus at the flatlands senate regarding the third dimension, A sphere takes A square back to Messiah INC in spaceland of which A Sphere is CEO. Full of questions and ideas, A square starts to ponder the idea of a fourth dimension in which A Sphere calls preposterous. With gravity taking its toll and things heating up politically in Spaceland due to A Squares, presence, A square is sent back to flatland where polygons are being trialed for suggestions of the third dimension. A square looks to defect to the North who are more liberal in their thinking. At the gates more altercations ensue, suddenly things start to disappear, sucked down through the fabric of Flatland until only A Square remains. He too starts disappearing until there is only his eye, and then a glowing point of light, welcoming him into another dimension. Due to our understanding of the first second and third dimensions and the way other dimensions interact with each other in the film, the movie leaves us with the feeling that a fourth dimension is not all that unreasonable, we just cant see it.

5. Flatlands from a Geometrical and Topological perspective

In order to understand topology from the perspective of flatlands, we need to look at how geometry and topology would work on a flatland, specifically how their intrinsic and extrinsic properties change. The best way to explain intrinsic and extrinsic properties is by using the mobius bank. Firstly lets imagine a rubber band in which a flatlander lives. Now imagine a spacelander has cut the band , put a twist in it and has reglued it. From a flatlanders perspective, he would witness the band being cut and being put back together. To him in flatland, the band hasn't changed just been cut and re-glued, as he cannot comprehend a twist in flatland. To us in spaceland however there is a clear twist in the band. Therefore we say that the intrinsic topology has remained the same as to flatlanders the shape is still the same, but the extrinsic topology has changed, as from a higher dimension the shape has changed.

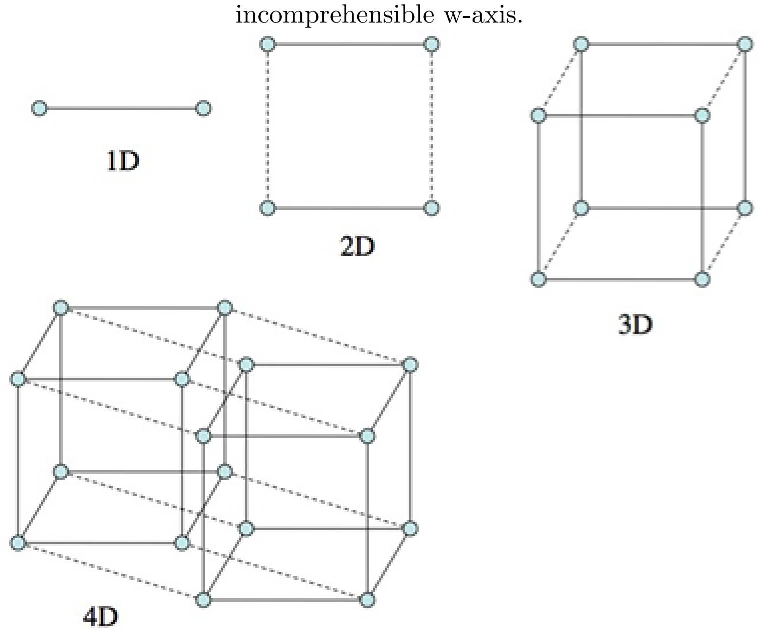


)

To see intrinsic or extrinsic properties geometrically, take a piece of paper and wrap it around a glass. The extrinsic geometry has changed however the paper itself hasn't been deformed, for example a flatlander wouldnt know its been curled and therefore its intrinsic geometry remains the same. Noting how intrinsic and extrinsic geometry's and typology's work are important for understanding manifolds and higher dimensions. We can curl , twist, connect parts of a 2D surface, and a flatlander living on it wouldn't notice it. This could be applied with a 4 dimensional person twisting , curling or connecting parts of our third dimension and we wouldn't even notice.

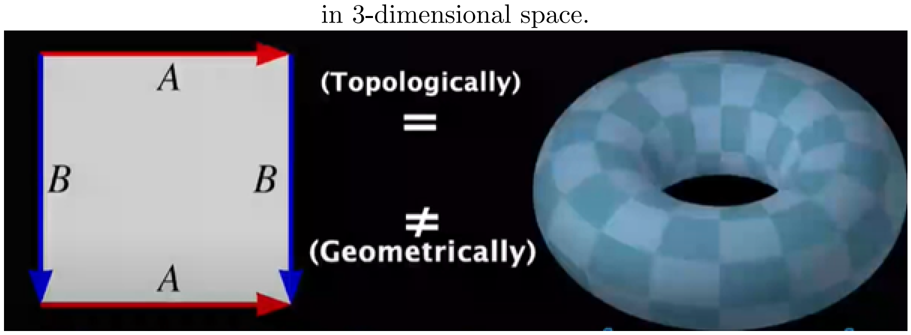
6. Visualising higher dimensions from a Geometrical Point of View

The idea of Flatland is extremely interesting, and not just to present the idea to the reader of a two dimensional being trying to perceive a three-dimensional world, but he enthusiastically keeps going where his main character 'A Square' postulates the existence of a fourth, and fifth, and an entire infinite sequence of dimensions. While living in a 3-dimensional world, we can easily visualize objects in 2 and 3 dimensions, however, mathematicians might consider this limiting. To geometrically understand the fourth dimension, we can try to explain shapes that exist in dimensions we can easily understand, then follow this understanding to the fourth dimension. A 1-dimensional line can be created by taking a point in the x-plane, making a copy, moving the copy a distance along the x plane, away from the original point, and connecting these points with a line. Similarly, a 2-dimensional square can be created in the xy-plane by making a copy of this line and moving it along the y axis and connecting it to our original line. Continuing this pattern, the cube is formed in the xyz-plane by creating an identical square, moving it along the z axis, and connecting these squares. We can then continue this train of thought by hypothesising an xyzw-plane where we create an identical cube and move it along the w-axis and connecting them with four lines to create a hypercube, as shown in the diagram below. However, just like A-square who lives in a 2-dimensional world and could not properly visualise 3-dimensional shapes on the unfathomable z-axis, we live in a 3-dimensional world and cannot properly visualise the 4-dimensional shapes like the hypercube on this mystifying, incomprehensible w-axis.

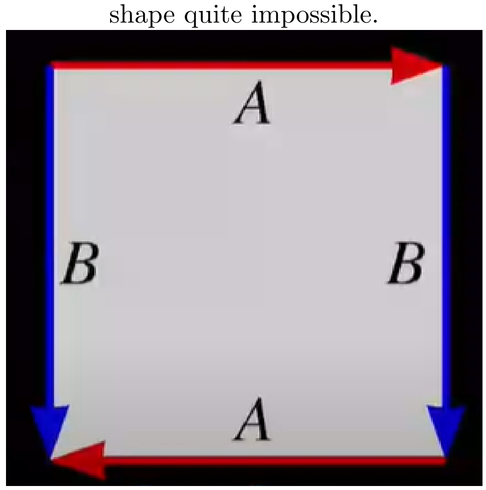


7. Higher dimensions from a Topological Perspective

To talk about higher dimensions from a topological perspective, we're going to talk about topological manifolds. A topological manifold is a locally Euclidean Hausdorff space. Given two distinct x,y in the topological space X, then X is a Hausdorff space if there exists disjoint sets U,V in X such that x is an element of U and y is an element of V. But at the most basic of levels, we can describe a manifold as the word a topologist uses for a shape. For example, a discrete space is a 0-dimensional manifold, a circle is a compact 1-manifold, a torus and a Klein bottle are compact 2-manifolds and an n-dimensional sphere S^n is a compact n-manifold. If we look at the torus as a surface, we can topologically visualize it as the square diagram on the left, we could see how Flatland's A-square would live in it. If he crossed the red or blue line, he would appear on the opposite side. Topology allows us to consider the torus in this way while geometry limits this understanding. A topologist would consider a torus as a 2-dimensional manifold in 3-dimensional space.



But what about this shape, where the direction of the arrows represent how the edges connect, where a 180 degree twist is needed to glue together both the A sides? If you tried this with a piece of paper, you would find putting together this shape quite impossible.



The above diagram actually represents the topology of a Klein bottle. But the only way to view this shape in 3-dimensions is through self-intersection as seen in the diagram on the next page..

As we try to draw a cube on the two-dimensional space of a piece of paper, we have to intersect lines to visualise it in three dimensions (as shown in the image under 'Visualising higher dimensions from a Geometrical Point of View'), where it's true form in three-dimensional space does not intersect. Similarly, as we try to draw a Klein bottle in three-dimensional space (in the form of an object), we have to allow for self-intersection to visualise it in four dimensions. In other words, while this object in the picture shows a clear self-intersection, if our brains could comprehend and sense the fourth dimension, we would not see any intersection at all. This is because a Klein bottle is a 2-dimensional manifold embedded in four-dimensional space. Topology is very useful for our understanding of higher dimensions as we try to see different manifolds in higher dimensions. A manifold exists in and of itself. It does not need to live in a higher-dimensional space.

8. Conclusion

In conclusion, although we have had great introduction to geometry throughout our education, topology could be called 'The Geometry of the Future'. Topology is an exciting field of mathematics, giving us the opportunity to discover exciting new facts about reality. Abbott's book 'Flatland: A Romance of Many Dimensions' gives us a deeper sense of understanding how humans could conceptualize a higher dimension, or how a lower dimensional being could conceptualize our dimension. With this understanding, we realize there is far more to reality than the ability of human perception. We may not be able to see many uses of topology for the near future, but I think that is what makes the study of topology so interesting. We can use this maths to, not only gain an understanding of higher dimensions, but to gain a better understanding of existence and reality as we know it. We are a species on an endless search for the ultimate understanding of 'being', always motivated by the lack of information known about the unknown. Topology is a mathematical field that helps us to open a door to see a glimpse of what we do not know, engendering an excitement like no other, the excitement of discovery and explanation of our reality.

References

[1] Nicholas Cooper (2019), *What is the difference between topology and geometry?*, Quora
[2] Edwin Abbot Abbot (1884), *Flatland: A Romance of Many Dimensions*, Seeley and Co.
[3] Shape of Space (nd), *Topology vs. Geometry*, Austin Community College District
[4] Thomas F Banchoff (1991), *Flatlands A New Introduction*, Princeton Science Library

[5] Anika Ridya-Dixit (2017), *Visualizing the Fourth Dimension*, Duke Research Blog
[6] Hong Lam (2020), *Topology and higher dimensions*, available on youtube

Email address: d.cassidykillane1@nuigalway.ie

Email address: c.forde13@nuigalway.ie

Email address: l.luptonsmith1@nuigalway.ie

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

NEIGHBOURHOODS AND OPEN SETS

ANTON SOHN AND SAMUEL HUDSON

(Communicated by David Futer)

ABSTRACT. Although a topology is most commonly defined in terms of opens sets, there is an equivalent definition arising from sets called neighbourhoods. It is quiet fiddly, which is why the much more useful open sets definition came about, but it is still valuable as a tool for understanding the concept of a topological space more intuitively. This Paper describes this definition, establishes its equivalence to the open sets definition, and compares the two definitions.

1. NEIGHBOURHOODS

Definition 1.1. We ask for a set X and for each point $x \in X$ a nonempty collection of subsets of X, called *neighbourhoods* of x. These neighbourhoods are required to satisfy the following axioms:

(a) x lies in each of its neighbourhoods.
(b) The intersection of two neighbourhoods of x is itself a neighbourhood of x.
(c) If N is a neighbourhood of x and if H is a subset of X which contains N, then H is a neighbourhood of x.
(d) If N is a neighbourhood of x and if $\mathring{N}$ denotes the set $\{z \in N | N \text{ is a neighbourhood of } z\}$, then $\mathring{N}$ is a neighbourhood of x. (The set $\mathring{N}$ is called the interior of N.)

The assignment of a collection of neighbourhoods satisfying these axioms to each point in X is called a *topology* on the set X. [1]

This is Felix Hausdorff's original definition of a topological space, [2] and was motivated by the search for an intuitive description of continuity that doesn't use a distance function.

What these axioms try to capture are the sets of points that in some way "surround" a given point x, without having to rely on a distance function. If a set is a neighourhood of a point x then we can take a small step in any direction starting from x and still be in the set. In our intuition there is a very close connection between the points that surround x and the distance from x to these points, so this idea is hard to grasp. Because of this I will give an example of a topology that *does* rely on distance to determine neighbourhoods, so that we are in familiar territory where we can understand the idea of a neighbourhood, but keep in mind that what

Received by the editors 3/6/21.
2010 *Mathematics Subject Classification.* 54-01.

the above definition is really trying to do is rid us of our dependence on distance.

A set N of $\mathbb{E}^n$ which contains a point x is called a neighbourhood of x if there exists a closed disc centred on x, which is contained entirely within N.[1] Note that this property need not hold for every point in N, only x itself.

It is easily checked that this definition of a neighbourhood follows the axioms above, and so it leads to a topology, which is called the *usual topology* on $\mathbb{E}^n$. The part of this definition that relies on a distance function is the closed disc, as it is defined to be the set of points no more than a certain distance from x. What we will find is that although it comes from a seemingly different place, this is the same usual topology on $\mathbb{E}^n$ that is defined in terms of which sets are deemed to be open.

2. The Open Sets Definition

Below is the more common definition of a topology.

Definition 2.1. A *topology* on a set X is a nonempty collection of subsets of X, called open sets, which are required to satisfy the following axioms:

(i) Any union of open sets is open
(ii) Any finite intersection of open sets is open
(iii) Both X and $\emptyset$ are open
[1]

Comparing this to definition 1.1, Notice that what we have gained in compactness and elegance we have lost in intuition. This set of axioms will make our lives much easier when carrying out complicated proofs, but someone who is learning topology for the first time might wonder where they came from, or what they represent.

The reason that these two definitions of topology can exist side by side is because they are equivalent. Roughly, this means that they always result in identical collections of open sets and assignments of neighbourhoods, given that open sets and neighbourhoods are defined in terms of each other in the right way.

3. Equivalence

Proposition 1. *Definitions 1.1 and 2.1 are equivalent*

To avoid confusion in the below proof I will use silly names to distinguish between the open sets we are given and the open sets we construct, and similarly for neighbourhoods. I will also refer to an assignment of a collection of neighbourhoods satisfying 1.1(a)-(d) to each $x \in X$ a one-topology on X and call a collection of open sets of X satisfying 2.1(i)-(iii) a two-topology on X.

Proof. Suppose we have a set X, and some one-topology on X.

We can construct a two-topology on X as follows; A set $O \subseteq X$ is open if it is a neighbourhood of each of its points. Lets check that our collection of open sets satisfies 2.1(i)-(iii).

(i) Suppose $\{O_i\}_{i \in I}$ is a (possibly infinite) collection of open sets.
Let $x \in \bigcup_{i \in I} O_i$. Then $x \in O_k$ for some $k \in I$ so O_k is a neighbourhood of x.

Clearly $O_k \subseteq \bigcup_{i\in I} O_i$ so by 1.1(c) $\bigcup_{i\in I} O_i$ is a neighbourhood of x. Therefore $\bigcup_{i\in I} O_i$ is open. ✓

(ii) Suppose $\{O_i\}_{i=1}^n$ is a finite collection of open sets.
Let $x \in \bigcap_{i=1}^n O_i$. Then $x \in O_k$ for all $1 \leq k \leq n$ so every O_k is a neighbourhood of x. By repeated application of 1.1(b) we find that $\bigcap_{i=1}^n O_i$ is a neighbourhood of x. Therefor $\bigcap_{i=1}^n O_i$ is open. ✓

(iii) Let $x \in X$. x must have some neighbourhood N, which must be a subset of X, so by 1.1(c), X is a neighbourhood of x. Therefore X is open. $\emptyset$ is vacuously a neighbourhood of each of its points. Therefore $\emptyset$ is open. ✓

Now, we can construct a one-topology on X, using these open sets, that is identical to our original one-topology, as follows.

Given $x \in X$ we call a set $M \subseteq X$ a *meighbourhood* of x if there exists an open set O such that $x \in O \subseteq M$.

Let $x \in X$ and let $M \subseteq X$ be a meighbourhood of X. Then there exists an open set O such that $x \in O \subseteq M$. But O is a neighbourhood of x since it is a neighbourhood of all of its points. So by 1.1(c) M is a neighbourhood of x.

Let $x \in X$ and let $N \subseteq X$ be a neighbourhood of X. By 1.1(d), $\mathring{N} \subseteq N$ is a neighbourhood of each of its points. Also $\mathring{N}$ is a neighbourhood of x by 1.1(d) so $x \in \mathring{N}$ by definition of $\mathring{N}$. We have an open set $\mathring{N}$ such that $x \in \mathring{N} \subseteq N$ so N is a meighbourhood of X.

We can conclude that given $x \in X$ and $N \subseteq X$, N is a neighbourhood of x if and only if it is a meighbourhood of X. So we have have constructed an assignment of a collection of "meighbourhoods" to each $x \in X$ which is identical to our original one-topology.

Now suppose we have a set X and some two-topology on X.

We can construct a one-topology on X as follows; Given $x \in X$, a set N is a neighbourhood of x if we can find an open set O such that $x \in O \subseteq N$. Lets check that these neighbourhoods satisfy 1.1(a)-(d).

(a) Suppose $x \in X$ and M is a neighbourhood of x. Then there exists an open set O such that $x \in O \subseteq M$. ✓

(b) Suppose $x \in X$ and M_1, M_2 are neighbourhoods of x. Then there exists open sets O_1, O_2 such that $x \in O_1 \subseteq M_1$ and $x \in O_2 \subseteq M_2$. Clearly $x \in O_1 \cap O_2$ and $O_1 \cap O_2 \subseteq N_1 \cap N_2$. Also, $O_1 \cap O_2$ is open by 2.1(ii) so $N_1 \cap N_2$ is a neighbourhood of x. ✓

(c) Suppose $x \in X$ and M is a neighbourhood of x. Then there exists an open set O such that $x \in O \subseteq M$. Let H be a subset of X which contains M. $x \in O \subseteq M$ so $x \in O \subseteq H$. Therefore H is a neighbourhood of x. ✓

(c) Suppose $x \in X$ and M is a neighbourhood of x. Let $\mathring{M}$ denotes the set $\{z \in M | M \text{ is a neighbourhood of } z\}$.

Now, we can construct a two-topology on X, using these neighbourhoods that is identical to our original two-topology, as follows;

We call a set $U \subseteq X$ *upen* if it is a neighbourhood of each of its points.

Let $U \subseteq X$ be upen. U is a neighbourhood of each of its points, so for each $x \in X$ we can find an open set O_x such that $x \in O \subseteq U$. It is not hard to see that $U = \bigcup_{x\in U} O_x$. By 2.1(i) U is open.

Let $O \subseteq X$ be open. For each $x \in O$, O is an open set such that $x \in O \subseteq O$. So O is a neighbourhood of each of its points so it is upen.

We can conclude that given $O \subseteq X$, O is open if and only if it is upen. So we have constructed a collection of "upen" sets which is identical to our original two-topology.

$\square$

In the language of category theory, definitions 1.1 and 2.1 both define concrete categories, and we have shown these categories to be concretely isomorphic. This means that we can regard the definitions as equivalent.

We can now take definition 1.1 to be our definition of a topology and define open sets as follows

Definition 3.1. We call a set $U \subseteq X$ *open* if it is a neighbourhood of each of its points. [1]

But we can equally well take definition 2.1 to be our definition of a topology and define neighbourhoods as follows

Definition 3.2. Given $x \in X$ we call a set $M \subseteq X$ a *neighbourhood* of x if there exists an open set O such that $x \in O \subseteq M$. [1]

Out of context these definitions might make it seem like open sets and neighbourhoods are defined circularly, but really we only every consider one of 3.1 or 3.2 at a time.

What we have shown above is that we can design any topology using either definition and it will result in the same structure if done correctly. So given definition 1.2, it is not hard to show that our description of the usual topology from section 1 is equivalent to the following.

A set O of $\mathbb{E}^n$ is open if for each $x \in O$ there exists a positive real number e such that the ball with centre x and radius e lies entirely in O. [1]

It is amazing that we can rephrase the definition of such a complicated structure like this, because different definitions may have different uses. Definition 1.1 is what formal topology was originally built on and is still a useful tool for students to develop a better understanding of topology, but 2.1 is much more practical for difficult mathematics. There are several other equivalent definitions of a topological space which may offer different insights or methods for solving a problem, which we can take advantage of.

References

1. Basic Topology - M. A. Armstrong
2. Characterizations of the category of topological spaces

18464724
Email address: `a.sohn1@nuigalway.ie`

18467574
Email address: `s.hudson3@nuigalway.ie`

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

THE TOPOLOGY OF ILLUSION: A STUDY IN TOPOLOGY-DISTRUBING OBJECTS

ETHAN GOODFELLOW AND EOGHAN BREHENY

(Communicated by Ethan Goodfellow, Eoghan Breheny)

Abstract. Recent Studies – e.g. (Sugihara, 2018b) – have brought to light a new class of mathematical objects which appear to disturb the topological properties when viewed from certain perspectives. Sugihara calls this phenomenon "topology-disturbing", as it seemingly defies the fundamental topologies of geometrical objects. We introduce the idea of these objects, explain how they work, present examples, and summarise the topological principles from which they are built.

Introduction

Evan Harris Walker asserts that optical illusions "constitute errors or limitations that accompany the data reduction processes involved in pattern discrimination. As a consequence, they provide valuable clues to the processes by which the data stemming from neural excitations are handled by the central nervous system to produce a discriminatory response to the optical pattern" (Walker, 1973).

Intuitively, the idea of topologically constructing an optical illusion seems complex. From science fiction[1] movies to M. C. Escher's House of Stairs (1951), optical illusions have an uncanny ability to impose a sense of impossibility on the observer. However, their mathematical derivation can often be straightforward. An important trait of the mathematical language is that it does not speak to our senses; as suggested by Walker, the structural triumph of illusion is rather a structural failure in perception.

1. Topology-Disturbing Objects

Topology-disturbing objects are a relatively new class of 3-dimensional objects, discovered by Kokichi Sugihara, which have an illusory quality when viewed from two specific perspectives. Namely, this quality is the perception of two or more objects being separate from one viewing direction, and connected or intersecting in another. These perspectives can be viewed simulataneously using a mirror, or sequentially with 180° rotations. Sugihara suggests that this perceptual phenomenon is likely "based on human preference and familiarity for canonical shapes such as a circle and a square. Strength of similar preference has been observed and studied

1991 *Mathematics Subject Classification.* Topology.

[1]A great example of a topological illusion can be seen in **this behind the scenes clip** from Nolan's "Inception" (2010), explaining how they simulated the famous Penrose Staircase, see (Penrose & Penrose, 1958).

in many contexts, including figure-ground discrimination, depth perception, visual search and line drawing interpretation. This is an important psychological aspect of the topology-disturbing objects" (Sugihara, 2018b).

2. Mathematical Derivation

2.1. **Ambiguous Cylinders.** Ambiguous cylinders are cylindrical objects that appear to have different structures when viewed from special directions (Sugihara, 2015). The mathematical principles used to construct such cylinders are important for creating topology-disturbing objects, and so we will briefly cover the mechanisms behind these objects.

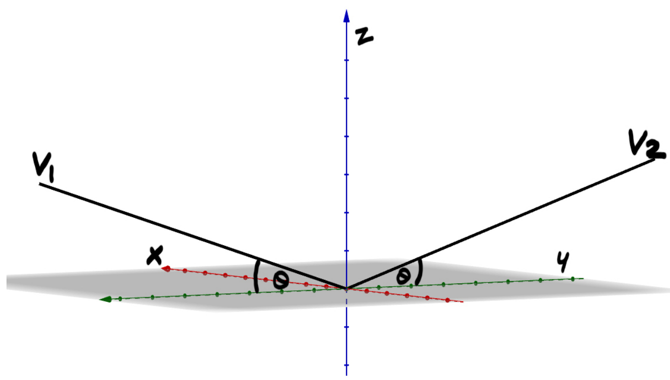


Figure 1. The Cartesian plane

Consider a typical xyz Carresian plane. We begin by positioning the xy-plane horizontally, and the positve z axis pointed upward. Given that these objects only appear to have different structures from two perspectives, we can denote these viewing directions as v_1 and v_2. We let these viewing directions be described by:

$$v_1 = (0, \cos\theta, -\sin\theta),$$
$$v_2 = (0, -\cos\theta, \sin\theta).$$

We then have our Cartesian plane and viewing directions as seen in Figure 1. Next, we fix two curves $a(x)$ and $b(x)$ for $x_0 \leq x \leq x_1$. Although these curves can be arbitrarily determined, it is important that the curves meet the following conditions: (1) both curves are x-monotone (i.e. $a(x)$ and $b(x)$ will not be self-intersecting), and (2) the initial and final x value must be equal, i.e. $a(x_0)$ and $b(x_0)$ are the initial points of the curve, and $a(x_1)$ and $b(x_1)$ are the final points. For each x, we consider the line passing through $a(x)$ from the perspective of, or parallel to, v_1 and similarly for $b(x)$ from the perspective of v_2. Therefore, these lines will lie on the same xy-plane, and thus intersect for each x. We can label this intersection as a third curve $c(x)$ – which forms a space curve, i.e. a curve which passes through any region of the xyz-plane. This is a difficult process to visualise abstractly, and so we have generated an illustration using two simple curves for $a(x)$ and $b(x)$ (see Figure 2 and 3).

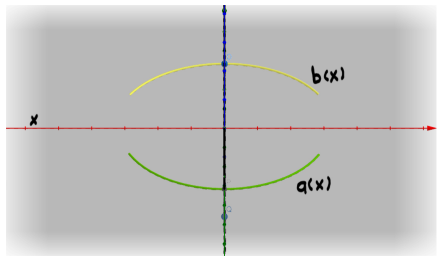


FIGURE 2. Perspective from v_1 with $a(x)$ parallel

Our space curve $c(x)$ then coincides with $a(x)$ when viewed from v_1, and coincides with $b(x)$ when viewed from $v2$. Next, we introduce a vertical line L which is kept vertical and its upper terminal point traces along $c(x)$ for $x_0 \leq x \leq x_1$. We build a surface S from the points swept by L which has the same length. Therefore, S will appear to be a cylindrical surface when viewed from v_1 and v_2 with a constant height. The upper edge of this surface appears as $a(x)$ when seen from v_1, and appears to be $b(x)$ when viewed from v_2. In actuality, the top edge is a non-uniform carving of $c(x)$. Because the surface is x-monotone it is not closed. Therefore, to construct a closed cylinder we need to apply this method twice (once for the upper half of the cylinder and another for the bottom half).

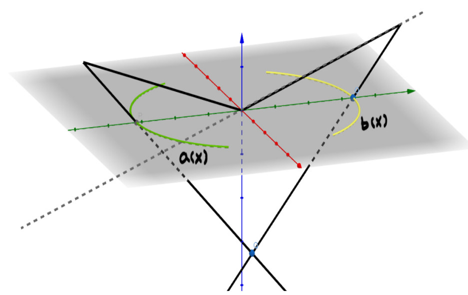


FIGURE 3. $a(x)$ intersects with $b(x)$ to create $c(x)$

2.2. **Construction.** Although examples of topology-disturbing objects can appear vastly different, their mathematical derivation employs the same methodology. There are nine[2] "generations" of what Sugihara terms impossible objects (i.e. mathematically constructed 3D optical illusions), but a common feature of the topology-disturbing generation is their v_1 perception of seperation and v_2 perception of connectedness (Sugihara, 2018a). Therefore, the construction of such objects is intuitively straightforward once the method of ambiguous cylinders is known. In a word, the unformal method is as follows: (1) Draw two or more separated objects, this will be our desired v_1 perspective, (2) Draw the same objects

[2]**See Webpage Here.**

with a symmetrical intersection (desired v_2 perception), (3) Decompose the result from (2) into non-intersecting closed curves, (4) Apply the method of ambiguous cylinders to the closed curves of (1) and (3), (5) Combine the resulting ambiguous cylinders such that neither v_1 or v_2 can percive the combining material, and also adjust their heights (explained below) to finalise the illusion. We will first provide an example of the construction of a topology-disturbing object before presenting the formalised topological theory behind them.

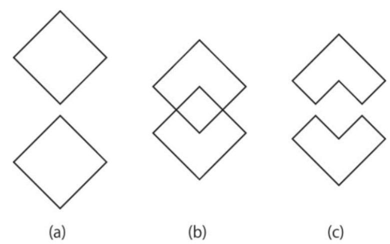


FIGURE 4.

For the sake of simplicity, we will illustrate an example of two topology-disturbing squares. We begin by creating three diagrams (Figure 4). The first, (a), shows our desired v_1 perspective of two separated squares. Diagram (b) shows our desired v_2 perspective whereby the squares are now connected and intersecting. (c) is our decomposition of (b) into two non-intersecting closed curves. Next, we apply the method of ambiguous cylinders (discussed in Section 2.1) twice, once for (a) and once for (c). We then have to connect these cylinders such that they appear to be separated from v_1 and intersecting in v_2. Notably, the step of adjusting the relative heights of both cylinders is because of the *intersecting* part of the illusion that is common to topology-disturbing objects; by moving the separate closed curves of (c) so that they are touching, they effectively appear as (b), which is why we must adjust the relative heights (to manipulate the perception of depth between v_1 and v_2 – See Figure 5).

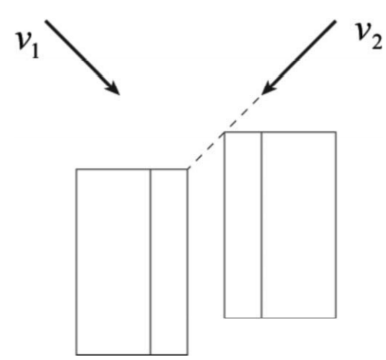


FIGURE 5.

We have now a 3-dimensional topology-disturbing object. Figure 6 illustrates the v_1 and v_2 (mirrored) perceptions that consistute the illusion, while Figure 7 shows the actuality of the 3D structure.

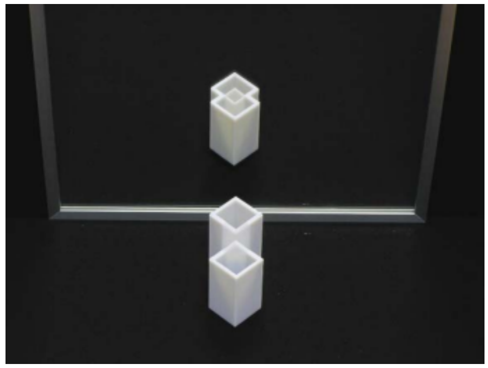


Figure 6.

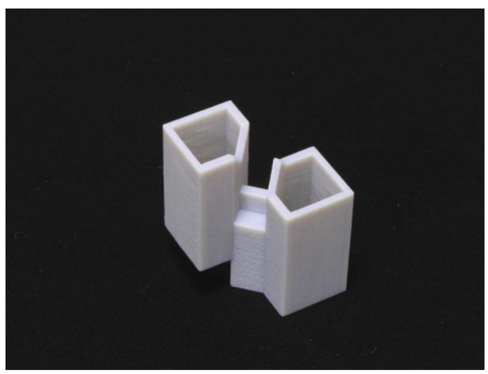


Figure 7.

3. Mathematical Principles

Now, we can provide the generalised theory behind the topology-disturbing objects. Let F be a line drawing composed of some curves drawn on the xy-plane such that each curve is a non-intersecting closed curve such as a circle/square or an open curve such as a line segment. We decompose the curves in F into x-monotone sgements and represent them by equations $y = f_1(x), y = f_2(x), ..., y = f_n(x)$ in such a way that

(1) two segments do not cross each other (they can still touch), and
(2) an upper segment has a smaller segment number than the lower segment, i.e. $f_i(x) \geq f_j(x)$ implies $i \leq j$.

Then, let G be another line drawing obtained from F by replacing each curve in the direction parallel to the y-axis, and $y = g_1(x), y = g_2(x), y = g_m(x)$ are x-monotone segments obtained by decoming G such that (1) and (2) are satisfied.

Then we can construct a topology-disturbing object whose appearances coincide with the drawings F and G, if the following conditions are satisfied:

(3) $m = n$, and
(4) for each i, $f_i(x)$ and $g_i(x)$ span the same x range, i.e. their leftmost points (initial points) have the same x-coordinate, and their rightmost points (end points) also have the same x-coordinate.

If these conditions are satisfied, then ambiguous cylinders can be constructed which correspond to $f_i(x)$ and $g_i(x)$ for $i = 1, 2, ..., n$. We denote these resulting ambiguous cylinders as $H_i, i = 1, 2, ..., n$. Next, we place the vertical cylinders $H_1, H_2, ..., H_n$ so that their appearance coincides with the drawings F and G when they are seen along the first view direction. If we see this collection of the cylinders along the second view direction, each cylinder has the desired appearance specified by the second line drawing G, but their mutual positions may not coincide with G. Therefore, as a final step, we translate the cylinders in the direction parallel to the first view direction so that the relative positions in the second appearance coincide with G. Then, we have built a topology-disturbing object.

References

Penrose, L. S., & Penrose, R. (1958). Impossible objects: A special type of visual illusion. *British Journal of Psychology*, *49*(1), 31–33.

Sugihara, K. (2015). Ambiguous cylinders: A new class of solid that evokes anomalous perception. *Journal of Vision*, *15*(12), 730–730.

Sugihara, K. (2018a). Evolution of impossible objects. In *9th international conference on fun with algorithms (fun 2018).*

Sugihara, K. (2018b). Topology-disturbing objects: a new class of 3d optical illusion. *Journal of Mathematics and the Arts*, *12*(1), 2–18.

Walker, E. H. (1973). A mathematical theory of optical illusions and figural aftereffects. *Perception & Psychophysics*, *13*(3), 467–486.

The Fields Medal: Topologist Winners

Eimear Burke and Brian Sweetman

Editor: Asamoah Nkwanta

Abstract

This paper will investigate the Fields Medal and some of the most famous winners of the award in the field of Topology and its related topics. A brief explanation will be outlined regarding what the Fields Medal is and we will then discuss the work of a few chosen mathematicians. The primary Mathematical Subject Classification (MSC) for this paper will be 54 General Topology.

1 Introduction

In this document you will read about the prestigious Fields Medal in mathematics with a particular focus on those who have won the award for their work in the field of Topology. We have looked at all of the winners from the past century and we have compiled a list of some of the winners who have a background in Topology. Some of the more interesting and relevant facts that we have come across from our research include the fact that there have been no Irish winners (as of May 2021). We have also discovered that there has only been one female winner of the award; Maryam Mirzakhani from Iran, and that was at the latest award ceremony in 2018. These points of interest will be further examined later in the document, along with some of the work from other winners of the award.

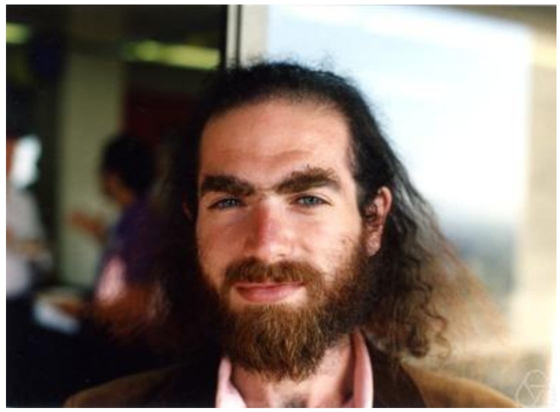
2 What is the Fields Medal

The Fields Medal is one of the most prestigious awards that can be won by mathematicians all around the world and is often described as the Nobel Prize of Mathematics. The award is presented at the opening ceremony of the International Congress of Mathematicians (ICM) every four years, which takes place in varying locations across the world wherein some of the brightest mathematicians and scientists gather for the special event. The first ceremony was held in Oslo, Norway in 1936. The most recent event occurred in Rio de Janiero, Brazil in 2018. Along with the actual medal, the winner receives a cash prize to the value of 15,000 Canadian Dollars.

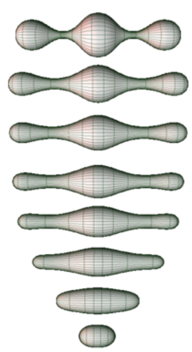
3 Previous Fields Medal Winners for Topology

Over the years there have been a number of recipients of the Fields Medal for their contributions to the subject of Topology. These include René Thom, John Milnor, Stephen Smale, Sergei Novikov, Simon Donaldson and Akshay Venkatesh. Of these, we will particularly focus on Grigori Perelman and Akshay Venkatesh and their work. In this way we will not only focus on someone who has achieved great things in Topology in the past and revolutionised it but we will also explore someone who has had one of the most recent developments in Topology.

4 Grigori Perelman and His Work



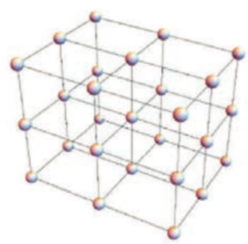
Grigori Yakovlevich Perelman was born on the 13th June 1966 in Leningrad, Soviet Union (now known as Saint Petersburg, Russia). Perelman is a Russian mathematician who is known for his contributions to the fields of Geometric Analysis, Riemannian Geometry, and Geometric Topology. In 2002 and 2003, Grigori Perelman presented a number of new results about the Ricci flow, including a novel variant of some technical aspects of Richard Hamilton's method. The Ricci flow is a certain partial differential equation for a Riemannian metric. It is an evolution equation which deforms Riemannian metrics by evolving them in the direction of minus the Ricci tensor. It is like a heat equation and tries to smooth out initial metrics(Chow et al., 2007). The image below illustrates several stages of Ricci flow on a 2D manifold.



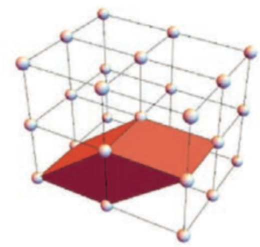
In August 2006, Perelman was offered the Fields Medal for "his contributions to geometry and his revolutionary insights into the analytical and geometric structure of the Ricci flow", but he declined the award and did not attend the International Congress of Mathematicians nor accept his prize money of 15,000 Canadian Dollars. He explained his reason for declining the Fields Medal by stating: "I'm not interested in money or fame; I don't want to be on display like an animal in a zoo." Perelman and Hamilton's works are now widely regarded as forming a proof of the Thurston conjecture, including as a special case the Poincaré conjecture, which had been a well-known open problem in the field of geometric topology since 1904. On 22nd December 2006, the scientific journal Science recognized Perelman's proof of the Poincaré conjecture as the scientific "Breakthrough of the Year", the first such recognition in the area of mathematics. On 18th March 2010, it was announced that he would receive the first Clay Millennium Prize for proving the Poincaré conjecture, now known as the Poincaré theorem. On 1st July 2010, he rejected the prize of one million dollars, stating: "I do not like their decision, I consider it unfair, I consider that the American mathematician [Richard] Hamilton's contribution to the solution of the problem is no less than mine." Many of Perelman's methods rely on a number of highly technical results from a number of disparate subfields within differential geometry, and as a result the full proof of the Thurston conjecture remains understood by only a very small number of mathematicians. However, the proof of the Poincaré conjecture, for which there are shortcut arguments due to Perelman and other mathematicians, is much more widely understood.

5 Akshay Venkatesh and His Work

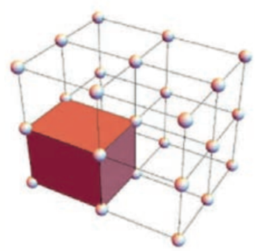
Akshay Venkatesh was born on the 21st November 1981 in New Dehli, India. He is of Australian Nationality however and is only the second Australian to have ever received the Fields medal. He is involved in many areas of mathematics but he has won the Fields Medal for the following: "His work uses representation theory, which represents abstract algebra in terms of more easily-understood linear algebra, and topology theory, which studies the properties of structures that are deformed through stretching or twisting, like a Möbius strip.". Below we will show some examples of his work in Topology. One of Venkatesh's most recent works is on the topic of locally symmetric spaces in six-dimensions. To describe this he asks us to picture a 3-D crystal. When we are given 3 side lengths and the angles they fall at, we can construct the cell of the crystal easily without any confusion on what it would look like. We can then place another cell on top of this to create the crystal. However, it is possible for various cells to create the same crystals. In order to make a crystal on a locally symmetric space we require 6 different coordinates and this in turn makes the locally symmetric space 6-Dimensional. See the diagram below for a visual representation of this.



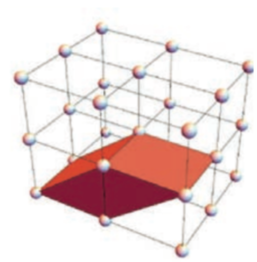
(a) crystal



(b) basic cell



(a) one basic cell



(b) different basic cell

6 Bibliography

En.wikipedia.org. 2021. Fields Medal - Wikipedia. [online] Available at: ¡https://en.wikipedia.org/wiki/Fields Medal¿ [Accessed 5 May 2021].

En.wikipedia.org. 2021. Akshay Venkatesh - Wikipedia. [online] Available at: ¡https://en.wikipedia.org/wiki/Akshay Venkatesh¿ [Accessed 5 May 2021].

En.wikipedia.org. 2021. Grigori Perelman - Wikipedia. [online] Available at: ¡https://en.wikipedia.org/wiki/Grigori Perelman¿ [Accessed 5 May 2021].

Venkatesh, A., 2018. What Information Might the Topology of Locally Symmetric Spaces Contain?. [online] Institute for Advanced Study. Available at: ¡https://www.ias.edu/ideas/venkatesh-locally-symmetric-spaces¿ [Accessed 5 May 2021].

Chow, B., Chu, S., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F. and Ni, L., 2007. The Ricci Flow: Techniques and Applications. Mathematical Surveys and Monographs,. [online] Available at: ¡https://www.ams.org/books/surv/135/surv135-endmatter.pdf¿ [Accessed 5 May 2021].

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

MA342 TOPOLOGY
KNOT THEORY

NIALL CARNEY 18302551 ENDA DALY 18357483

Abstract. The aim of this project is to explain in greater detail Knot Theory to fellow MA342 students. In order to achieve this goal, we shall discuss what is Knot Theory, what are the main goals associated with Knot Theory, and the origins behind it.

1. What is Knot Theory?

In mathematics, a knot is defined as a closed, non-self-intersecting curve that is embedded in three dimensions and cannot be untangled to produce a simple loop.

The simplest knot in mathematics, known as the trivial knot (unknot) can be shown by a simple rubber band. Our general idea of a knot is different to a mathematical knot. The knot you tie in your shoe laces is not a mathematical knot due to the fact it has loose ends. A mathematical knot must be a closed curve with no loose ends.

Different projections of the same knot can have a different number of crossings. For example the unknot can also be projected so that it has one crossing.

We say these two knots above are both the same knot as we can untangle one to create the other without ever letting the string pass through itself. Any kind of untangling of knots can be described by three moves which Reidemeister described as:

Received by the editors 06/06/21.
2010 *Mathematics Subject Classification.* Knot Theory .

Figure 1. What is a Knot?

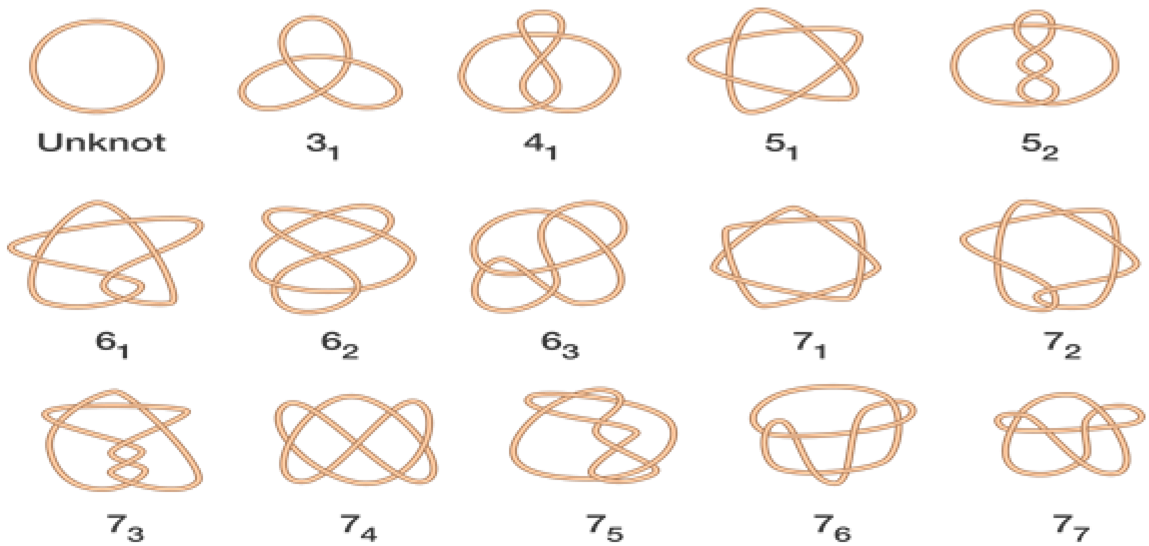


Figure 2. Examples of Knots

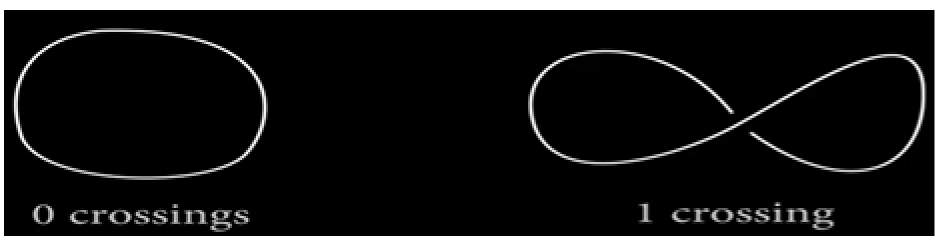


Figure 3. What is a Knot?

- Twisting
- Moving one string over or under another.
- Moving one string over or under a crossing.

2. What are the goals of Knot Theory

The original motivation for the founders of knot theory was to create a table of knots and links, which are knots of several components entangled with each other. More than six billion knots and links have been tabulated since the beginnings of knot theory in the 19th century.

Knot theory has uses in physics, biology and other fields. In the field of biology knot theory is used to provide an insight into how hard it is to unknot and reknot various types of DNA, shedding light on how much time it takes the enzymes to do their jobs.

Knot theory is also used in the very relevant world of cryptography. With the rapid rise in the popularity of cryptocurrencies and the role they might play in how we make payments in the future, knot theory plays a role in this. Knot theory is used in Algebraic Eraser, a security protocol that uses knots in cryptography. The protocol weaves mathematical information from knots into encryption keys that allow users to decipher secret data encoded in security badges, mobile payment devices and more.

3. History of Knot Theory

Knots first came to prominence in the 1860s, however there is evidence that Knot Theory was studied in prehistoric times. The emergence in the 19th century arose when scientists were trying to understand the nature of matter. At the time, people didn't know that atoms were nuclei surrounded by electrons, so Lord Kelvin proposed a different theory: He hypothesized that the basic building blocks of matter were knots in the ether, a hypothetical substance that permeated space. He postulated that every element – hydrogen, oxygen, gold and so on – was made from a different kind of knot.

As time passed, Knot Theory became a matter of interest in the mathematical field of Topology. 20th century topologists Max Dehn and J.W. Alexander were interested in Knot Group and invariant from homology theory. As a result, the Alexander polynomial was born. This creation would soon become the main avenue for studying Knot Theory for a period of time until ground breaking changes occured.

More recently, in the 1970's, the notion of hyberbolic geometry was introduced by William Thurston into Knot Theory. This soon led to the discovery that many knots were hyperbolic knots. From here, the discovery of Jones' polynomial and other contributions led to connections in knot theory and other fields of mathematics such as statistics and mathematical methods.

4. Bibliography

1. Wikipedia - Knot Theory
2. pi.math.cornell.edu - Knot Theory
3. maths.ed.ac.uk - KNOT THEORY and ITS APPLICATIONS

National University of Ireland, Galway

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

GEOMETRIC TOPOLOGY: LOW-DIMENSIONAL AND HIGH-DIMENSIONAL TOPOLOGY

ROBERT DEERY AND JAN KRUSZYŃSKI

(Communicated by The American Mathematical Society)

Abstract. In this paper we hope to provide a better understanding of geometric topology with reference to concepts in low-dimensional and high-dimensional topology. In particular, we first define topological manifolds; then explore the topics of the uniformization theorem, and surgery theory for the study of manifolds of dimension greater than three.

1. Topological Manifolds

1.1. **Definition.** A topological space X is called locally Euclidean if there is a non-negative integer n such that every point in X has a neighbourhood which is homeomorphic to real n-space $\mathbb{R}^n$. [1] A manifold is a topological manifold by definition and also is a locally Euclidean Hausdorff space. A manifold can take up additional requirements such as being smooth, paracompact, homological, second-countable, etc.

1.2. **Riemannian Manifolds.** Many classical theorems in Riemannian geometry show that manifolds with positive curvature are constrained. Conversely, negative curvatures are generic. For example, any manifold of dimension $n \geq 3$ admits a metric with the negative Ricci curvature. [2]

1.3. **Curvatures.** Using compact Riemann surfaces, a classification of closed orientable Riemannian 2-manifolds follows that each such is conformally equivalent to a unique closed 2-manifold of constant curvature, which is a quotient of a discrete subgroup of an isometry group [7]:

- Sphere (curvature +1)
- Hyperbolic plane (curvature -1)
- Euclidean plane (curvature 0)

Received by the editors 13/05/2021.
2020 *Mathematics Subject Classification.* Primary 57-00. Secondary 57K20, 57M10, 57M50, 57N45, 57N65, 57R40, 57R65, 57R67, 57R80.

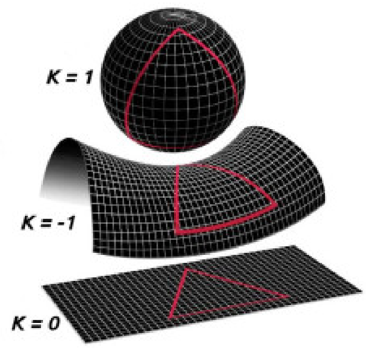


Figure 1. Positive, Negative and Zero Gaussian curvatures

1.4. **Classification of 2-manifolds.** The classification is consistent with the Gauss–Bonnet theorem, which implies that for a closed surface with constant curvature, the sign of that curvature must match the sign of the Euler characteristic. The Euler characteristic is equal to 2 – 2g, where g is the genus of the 2-manifold. [7]

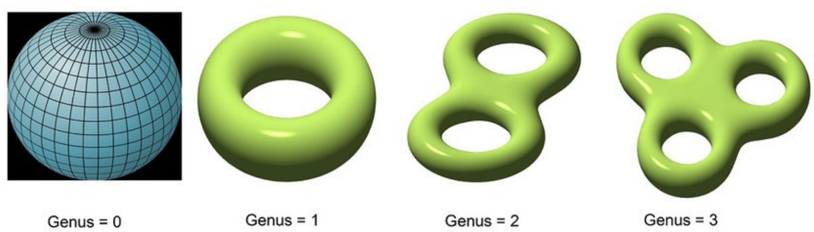


Figure 2. Examples of various 2-manifolds

2. Uniformization Theorem

2.1. **Definition.** The uniformization theorem says that every simply connected Riemann surface is conformally equivalent to one of three Riemann surfaces: the open unit disk, the complex plane, or the Riemann sphere. In particular it implies that every Riemann surface admits a Riemannian metric of constant curvature. [3] Also, as stated before, in terms of closed Riemannian 2-manifolds; each manifold has a accordingly equivalent Riemannian metric with a constant curvature.

2.2. **Maskit Uniformization theorems.**

Theorem 2.1. *Let S be an oriented surface and let $v_1 \dots v_n \dots$ be a set of pairwise disjoint loops on S. If $\tilde{S} \to S$ is a regular covering with defining subgroup $N = \langle v_1^{\alpha_1} \dots v_n^{\alpha_n} \dots \rangle$, where $\alpha_1 \dots \alpha_n \dots$ are natural numbers, then $\tilde{S}$ is a flat covering that is homeomorphic to a domain in $\overline{\mathbb{C}}$. [5]*

Theorem 2.2. *Let $\tilde{S}$ be a flat surface and let $\tilde{S} \to S$ be a regular covering of an oriented surface S with defining subgroup N. If S is a surface of finite type, i.e.*

$\pi_1(S)$ is finitely generated, then there exists a finite set of simple pairwise disjoint loops $v_1 \dots v_n$ and natural numbers $\alpha_1 \dots \alpha_n$ such that $\langle v_1^{\alpha_1} \dots v_n^{\alpha_n} \rangle = N$. [5]

Theorem 2.3. *If $\tilde{S}$ is a flat Riemann surface and $\overline{\mathrm{G}}$ is a properly-discontinued group of conformal automorphisms of $\tilde{S}$, then there exists a conformal homeomorphism $h : \tilde{S} \to D \subset \overline{\mathrm{C}}$ such that hGh^{-1} is a Kleinian group with invariant component D. [6]*

Corollary 2.3.1 (S in a closed Riemann surface of genus ≥ 1)**.** Thus, every Riemann surface is uniformized by a Kleinian group, then its fundamental group has the presentation

$$\pi_1(S) = \{a_1, b_1 \dots a_g, b_g : \prod_{j=1}^{g} [a_j, b_j] = 1\} \text{ [4]} \tag{2.1}$$

and the normal subgroup N defined by the flat covering $\tilde{S}$ may be taken to the smallest normal subgroup generated by $a_1 \dots a_g$(or $b_1 \dots b_g$); S is now uniformized by the Schottky group G of genus g- a free purely-Ioxodromic Kleinian group with g generators. [4]

3. Surgery theory

3.1. **Definition.** Surgery theory has its origins in geometric typology as a collection of techniques for turning one manifold into another, of the same dimension n. This involves cutting out parts of the manifold and replacing it with a part of another, along the boundary. This seemingly inconsequential operation not only has its uses in the field of differential manifolds but also extends its applications into such areas as piecewise linear (**PL-**) and topological manifolds (**TOP**), making it a valuable tool in the topologist's arsenal when working with manifolds of dimension greater than three.

3.2. **Surgery on a manifold.** The main idea of performing surgery on a manifold is to produce another manifold with some desired property, while keeping any relevant topological invariant(s) of the manifold known. To understand this, we'll take two manifolds X and Y with boundary and boundary product

$$\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y) \text{ [8]} \tag{3.1}$$

To justify surgery we need the following to hold:

$$\partial(S^p \times D^q) = \partial(D^{p+1} \times S^{q-1}) \text{ [8]} \tag{3.2}$$

where D^q is the q-dimensional disk and S^p denotes the p-dimensional sphere.

The effect of a surgery on $S^0 \times D^2$ is to attach a handle, whereas a surgery on $S^1 \times D^1$ removes a handle. In this way, surgery closely resembles the idea of handle attaching.

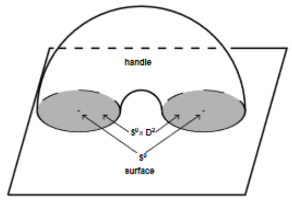


FIGURE 3. Surgery on an embedded $S^0 \times D^2$ [9]

Surgery produces a manifold M' [1] by cutting out $S^p \times D^q$ and gluing in $D^{p+1} \times S^{q-1}$, or by performing a p-surgery for a given p. So, given an M' with boundary $(L, \partial L)$ and an embedding $\phi : S^p \times D^q \to \partial L$ we can obtain another $(n+1)$-manifold with boundary L' by attaching a $(p+1)$-handle, with $\partial L'$ obtained from ∂L by a p-surgery:

$$\partial L' = (\partial L - int\ im\phi) \cup_{\phi|_{S^p \times S^{q-1}}} (D^{p+1} \times S^{q-1}) \text{ [8]} \tag{3.4}$$

The operation of surgery introduces the corresponding notion of a cobordism W between M and M', where the **trace** of the surgery is the cobordism:

$$W := (M \times I) \cup_{S^p \times D^q \times \{1\}} (D^{p+1} \times D^q) \text{ [8]} \tag{3.5}$$

3.3. **Classification of manifolds.** The fundamental and most important application of surgery theory is the classification of manifolds and manifold structures. The two key problems in determining the existence of manifold structures are as follows:

- Is X a manifold? In other words, does a space X have the homotopy type of a smooth manifold of the same dimension?
- Is f a diffeomorphism?[2] Does there exist a homotopic equivalence $f : M \to N$ between two smooth manifolds homotopic to a diffeomorphism?

The classification of manifolds does not provide a complete set of invariants $f_i : X \to Y_i$, but instead uses the **surgery obstructions**, which define a map $\theta : \mathcal{N}(X) \to L_n(\pi_1(X))$ from the **normal invariants** to the **L-groups**. An example of this is surgery performed on the ***normal maps***, which takes a normal map and produces another normal map within the same bordism class. This is outlined in the classical approach to surgery, as developed by Browder, Novikov, Sullivan and Wall, for normal maps of degree one.[8] The question becomes: "Is the normal map $f : M \to X$ of degree one cobordant to a homotopy equivalence?" To answer this question we use the algebraic statement about some element in an **L-group** of the **group ring** $\mathbb{Z}[\pi_1(X)]$. If the **surgery obstruction** $\sigma(f) \in L_n(\mathbb{Z}[\pi_1(X)])$ is zero

[1] A manifold M' is defined as follows

$$M' := (M \setminus int(im(\phi))) \cup_{\phi|_{S^p \times S^{q-1}}} (D^{p+1} \times S^{q-1}) \text{ [8]} \tag{3.3}$$

[2] A diffeomorphism is a map between manifolds which is differentiable and has a differentiable inverse. [10]

then the answer to the above question is yes.[3] So, a space X has the homotopy type of a smooth manifold if and only if it receives a normal map of degree one whose surgery obstruction vanishes.[8] The result of this analysis is best shown by the *surgery exact sequence* of Sullivan and Wall,

(3.6)
$$\ldots \longrightarrow \mathcal{N}_\partial(X \times I) \longrightarrow L_{n+1}(\pi_1(X)) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{N}(X) \longrightarrow L_n(\pi_1(X)) \text{ [11]}$$

which relates three different items:

(1) the *structure set* $\mathcal{S}(X)$ of the Poincaré complex X, which measures the number of distinct manifolds (up to the appropriate notion of isomorphism) in the simple homotopy class of X
(2) normal data $\mathcal{N}(X)$, which measures the possible characteristic classes of the normal or the tangent bundle of manifolds in the simple homotopy type of X
(3) the Wall surgery groups $L_n(\mathbb{Z}\pi)$, depending only on the fundamental group π of X and the dimension n (modulo 4). [9]

References

1. John M. Lee (6 April 2006). *Introduction to Topological Manifolds*, Springer Science & Business Media. ISBN 978-0-387-22727-6.
2. Gallot, Sylvestre; Hulin, Dominique; Lafontaine, Jacques (2004) *Riemannian geometry*, Universitext (3rd ed.), Berlin: Springer-Verlag.
3. Abikoff, William. *The Uniformization Theorem.* The American Mathematical Monthly, vol. 88, no. 8, 1981, pp. 574–592. JSTOR, `www.jstor.org/stable/2320507`. (Accessed 8 May 2021).
4. Gusevskii, N.A. (2001) [1994], "Uniformization", Encyclopedia of Mathematics, EMS Press
5. B. Maskit, "A theorem on planar covering surfaces with applications to 3-manifolds" Ann. of Math. , 81 : 2 (1965) pp. 341–355
6. B. Maskit, "The conformal group of a plane domain" Amer. J. Math. , 90 : 3 (1968) pp. 718–722
7. 'Uniformization theorem' (2021). *Wikipedia*. Available at: `https://en.wikipedia.org/wiki/Uniformization_theorem` (Accessed 7 May 2021).
8. 'Surgery Theory', (2021). *Wikipedia*. Available at: `https://en.wikipedia.org/wiki/Surgery_theory` (Accessed: 24 April 2021).
9. Rosenberg, J., 2014. Surgery theory today—what it is and where it's going. *Surveys on Surgery Theory*, 2, pp.3-47.
10. Weisstein, Eric W. "Diffeomorphism." From *MathWorld*-A Wolfram Web Resource. Available at: `https://mathworld.wolfram.com/Diffeomorphism.html`
11. 'Surgery Exact Sequence' (2021). *Wikipedia*. Available at: `https://en.wikipedia.org/wiki/Surgery_exact_sequence` (Accessed: 1 May 2021).

[3] A normal map of degree one is cobordant to a homotopy equivalence if and only if the signatures of domain and codomain agree [8]

FELIX KLEIN AND THE KLEIN BOTTLE

RONAN FINNEGAN AND JACK SULLIVAN

Abstract. This is a brief glimps into the life of Felix Klein and the Klein bottle. Also included is how to construct a klein bottle from two mobius strips.

Contents

1. Introduction

Figure 1. Felix Klein

Born on 25 April 1849 in Düsseldorf, Felix Klein went on to be a very successful mathamatician. He is remebered most for his work in topology when he discovered the aptly named Klein bottle. Klein studied maths and physics at the University of Bonn from 1865-1866. He recieved his doctorate under Julius Plücker in 1868. Upon Plückers death in 1868 Klien continued to write the second half of the book Neue Geometrie des Raumes. 1872 Klein was appointed professor in the University of Erlangen. He left for another professorship in 1875 at Technische Hochschule München. After 5 years Klein was appointed to a chair of geometry at Leipzig. [1]

2. Work

Klein has a lot of notable work in Mathamatics and Physics. Klein's dissertation, on line geometry and its applications to mechanics, classified second degree line complexes using Weierstrass's theory of elementary divisors. Some of his other work includes 'On the So-called Euclideean Geometry'. In this book Klein influenced the 'Erlangen program'. Klein also worked on complex analysis, specificly the link between Riemann and invariant theory, number theory, group theory and complex geometry.[1]

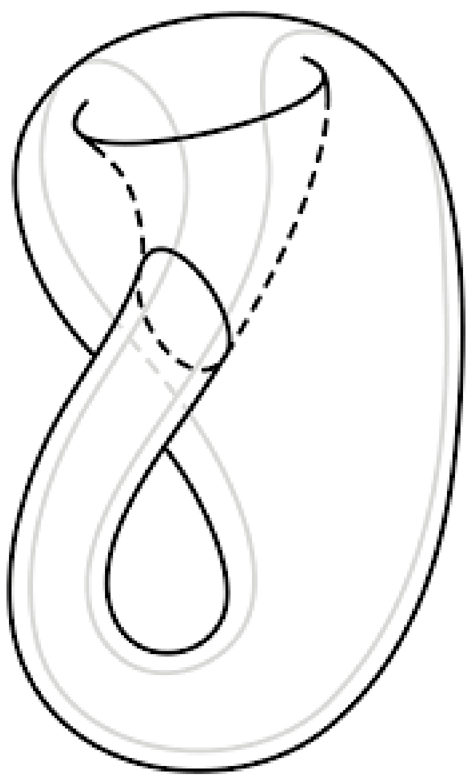


Figure 2. Klein Bottle

3. Klein Bottle

The Klein bottle may not be Felix Klein's most profound discovery but it is certainly his most widely recognized. It was first described by Klein in 1882 while he was tenuring in Leibzig University, Germany. It is a surface with very interesting properties. The first is that it is a closed surface; this means that something travelling along the face of the Klein bottle could go in any direction for as long as they wanted without ever encountering a boundary or an edge. A sphere is an example of another closed surface. But what differenciates it from a sphere is that the Klein bottle only has one side.As with a mobius strip, you could paint the entirety of its surface without lifting your paintbrush. The technical term for this type of surface is a non-orientable manifold.

3.1. **Construction.** An example of a mobius strip can be formed by taking a strip of paper, twisting one side 180 degrees and then gluing the edges together.

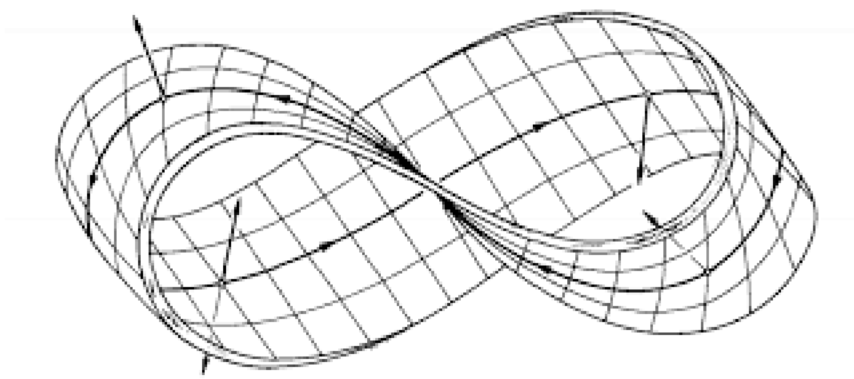


FIGURE 3. The mobius strip

This surface has only one edge and side and is called a mobius strip.Any topological space homeomorphic to this example is also called a mobius strip.

The Klein bottle can then be formed by gluing the edges of two mobius strips (one created with a left handed twist and the other with a right handed twist) together. This leaves a surface with no edges and only one face, the Klein bottle. Another, but equilvalent, way to think about constructing a Klein bottle is by using

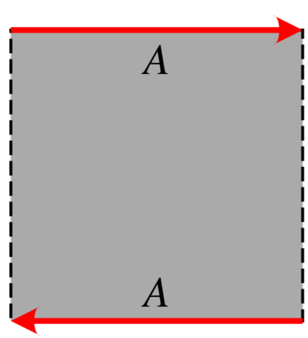


FIGURE 4. The fundamental polygon of a mobius strip. The mobius strip can be formed by attaching side A to other side A so that the red arrows point in same direction.

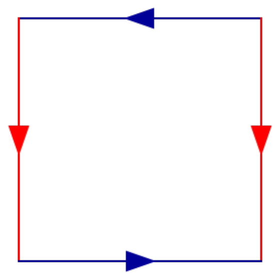


FIGURE 5. The fundamental polygon of Klein bottle. It is A 3-d representation of a Klein bottle with self constructed by attaching the red and blue arrows intersection. to the same colour respectively so that each pair of arrows face the same way.

the fundamental polygon. This is similar to the fundamental polygon of the mobius strip above, but with the added condition of attaching the “left over“ sides.

The Klien bottle can then be formed by gluing the edges of two mobius strips (one created with a left handed twist and the other with a right handed twist) together. This leaves a surface with no edges and only one face, the Klein bottle. Another, but equilvalent, way to think about constructing a Klein bottle is by using the fundamental polygon. This is similar to the fundamental polygon of the mobius strip above, but with the added condition of attaching the “left over“ sides.

The Klien bottle can then be formed by gluing the edges of two mobius strips (one created with a left handed twist and the other with a right handed twist) together. This leaves a surface with no edges and only one face, the Klein bottle. Another, but equilvalent, way to think about constructing a Klein bottle is by using the fundamental polygon. This is similar to the fundamental polygon of the mobius strip above, but with the added condition of attaching the “left over“ sides.

If we were to try to use these methods to create a Klein bottle in our 3 dimensional world we would get a self intersection at some point on the surface. This is because it is impossible to create a true, non-self intersecting Klein bottle in less than 4 dimensions. The extra dimension preventing self intersection can be explained through the use of an analogy. Imagine looking at a loop of string from above and viewing it as a 2-d image. What we would see is a self intersecting piece of string at the points where the ends cross over each other. When viewing from a 3-d reference however we can see that no self intersection occurs and simply one piece of string is above the other.

This is analogous to our 3 dimensional implementation of the Klein bottle which creates self- intersection, which could be avoided if we could percieve 4 spatial dimensions.[2]

References

[1] Felix Klein'(2021).*Wikipedia*. Available at: https://en.wikipedia.org/wiki/Felix_Klein (Accessed:01 June 2021)

[2] Klein Bottle'(2021).*Wikipedia*. Available at: https://en.wikipedia.org/wiki/Klein_bottle (Accessed 01 June 2021

National University of Ireland Galway, Ireland
Email address: ronanfinnegan1@gmail.com

National University of Ireland Galway, Ireland
Email address: swiftjsulli@gmail.com

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

$\sim\approx$

FIELDS MEDAL WINNERS FROM THE FIELD OF TOPOLOGY

JAMES FOLEY, DIARMUID DONNELLAN AND JOHN TIERNEY

Abstract. While physicists, chemists, and economists dream of winning the Nobel Prize, arguably the greatest honour a mathematician can achieve is winning the Fields Medal. In this paper we will explain what the Fields Medal is and why it is so lucrative in the world of Mathematics. We will then give a brief account of four recipients of this medal who contributed to the field of Topology, and what their contributions were that led them to wining this award. The four topologists we chose were Sergei Novikov, William Thurston, Stephen Smale, and Vaughan Jones.

1 What is the Fields Medal?

Figure 1. Fields Medal

1.1 The Fields Medal is a prize awarded every four years at the International Congress of Mathematicians. The medal recognises outstanding achievements in Mathematics. The Executive Committee of the International Mathematical Union honour between two and four mathematicians under the age of 40. Along with the Abel Prize, it is seen as the Mathematics equivalent to the Nobel Prize [11]. In addition to the medal, recipients are awarded a cash prize of CAD 15000. The medals and cash prizes are funded by a trust established by J.C.Fields at the

2010 *Mathematics Subject Classification.* Primary .

University of Toronto. In 2014, Maryam Mirzakhani made history by becoming the first female recipient of the award [9]. The most recent winners were Peter Scholze, Alessio Figalli, Akshay Venkatesh, and Caucher Birkar [10]. While the prize can be awarded for contributions to differential equations, algebraic geometry, and dynamic systems among others, this paper will focus on medalists who made contributions towards the field of Topology.

2 Sergei Novikov.

Figure 2. Sergei Novikov

2.1 **Background.** Sergei Novikov was born on 20 March 1938 in what is now known as Nizhny Novgorod in the Soviet Union. Novikov's was born into a family of talented mathematicians, with his father Pyotr Sergeyevich Novikov giving a negative solution to the word problem for groups. In 1960 Novikov graduated from Moscow State University before achieving a Ph.D and a Doctor of Science degree from the V.A. Steklov Institute of Mathematics in Moscow. In 1983 he became head of the mathematics department at the Steklov Institute. Novikov is celebrated as one of the greatest mathematicians of the 20th century. In addition to his Fields Medal, he also received the Moscow Mathematical Society Award for young mathematicians, the Lenin Prize and the Wolf Prize. Novikov made many contributions to the field of Topology, specifically in algebraic topology and differential topology. In his later years he attempted to find links between theoretical physics and modern mathematics, particularly in solitons and spectral theory. [3] [12]

2.2 **The Novikov Conjecture.** The Novikov Conjecture attempts to answer the question "Which expressions of the rational Pontryagin characteristic classes are

homotopy invariant for the closed manifolds and how should we classify them?". As of yet the conjecture is unsolved for all groups yet has been proved for finitely generated abelian groups. According to the Novikov conjecture, the
higher signatures, which are certain numerical invariants of smooth manifolds, are homotopy invariants. Here we let G be a discrete group, BG its classifying space and M a closed oriented n-dimensional manifold. Also important to note is that BG is is an Eilenberg–MacLane space of type K(G,1), and therefore unique up to homotopy equivalence as a CW complex.
Let $f : M \to BG$ be a a continuous map from M to BG, and $x \in H^(n-4i)(BG;\mathbb{Q})$. Novikov considered the numerical expression called the higher signature: $\langle f^*(x) \cup L_i(M), [M]\rangle \in \mathbb{Q}$. Novikov found this by evaluating the cohomology class in top dimension against the fundamental class $[M]$. In the expression for the higher signature, L_i is the i^{th} Hirzebruch polynomial. For each i, this
polynomial can be expressed in the Pontryagin classes of the manifold's tangent bundle. The Novikov conjecture states that the higher signature is an invariant of the
oriented homotopy type of M for every such map f and every such class x, in other words, if $h\colon M' \to M$ is an orientation preserving homotopy equivalence, the higher signature associated to $f \circ h$ is equal to that associated to f. As of 2021, no
counterexamples have been offered to the conjecture, and it remains one of the most important unsolved problems in Topology. For this conjecture, Sergei Novikov was awarded the Fields Medal in 1970. [7]

2.3 **Novikov's Other Contributions to Topology.** Novikov also made other very important contributions to Topology, most notably in corbodism theory and in geometric topology. Early in his career he showed how the Adam's spectral sequence could be adapted to the new (at that time) cohomology theory typified by cobordism and K-theory. This led to the development of the idea of cohomology operations in the general setting.
Novikov also examined foliations - the decomposition of manifolds into smaller ones called leaves. At the time it was not known whether leaves could be closed, but Novikov's proof of the existence of closed leaves in the case of a three-sphere led to a lot of major contributions to the field. [3]

2.4 **Later Work.** Novikov's reputation as a famous mathematician grew in his later years. He was elected a Full Member of the USSR Academy of Sciences in 1981, and in 1984 he was elected as a member of Serbian Academy of Sciences and Arts. He has also worked as a distinguished professor and researcher in various Russian and American universities. As recently as 2020, he received the Lomonosov Gold Medal at the Russian Academy of Sciences. [3]

3 William Thurston.

3.1 **Background.** William Paul Thurston, born October 30th , 1946 in
Washington, D.C., U.S.A and later died on August 21st , 2012 in Rochester, New York was an American mathematician who won the 1982 Fields Medal for his work in topology. Having been educated at the New College, Sarasota, Florida where he received his B.A. in 1967, he went on to study at the University of California, Berkeley where he received his Ph.D. in 1972. After a year at the Institute for Advanced Study, Princeton, New Jersey, he joined the faculty of the Massachusetts

Institute of Technology from 1973 to 1974 and he then moved to Princeton University, where he remained until 1991. In 1992 he became director of the Mathematical Sciences Research Institute at Berkeley. Thurston later taught at the University of California, Davis from 1996 to 2003 and later at Cornell University from 2003 to 2012.

3.2 **Fields Award and later work.** At the International Congress of Mathematicians in Warsaw in 1983 he received the Field's award for his work in the topology of two and three dimensions, showing interplay between analysis, topology, and geometry. He also contributed the idea that a very large class of closed 3-manifolds carry a hyperbolic structure. The headline behind Thurston winning this award was as follows,
"Revolutionized study of topology in 2 and 3 dimensions, showing interplay between analysis, topology, and geometry. Contributed idea that a very large class of closed 3-manifolds carry a hyperbolic structure."
Thurston went on to take up ideas about the discrete isometry groups of hyperbolic

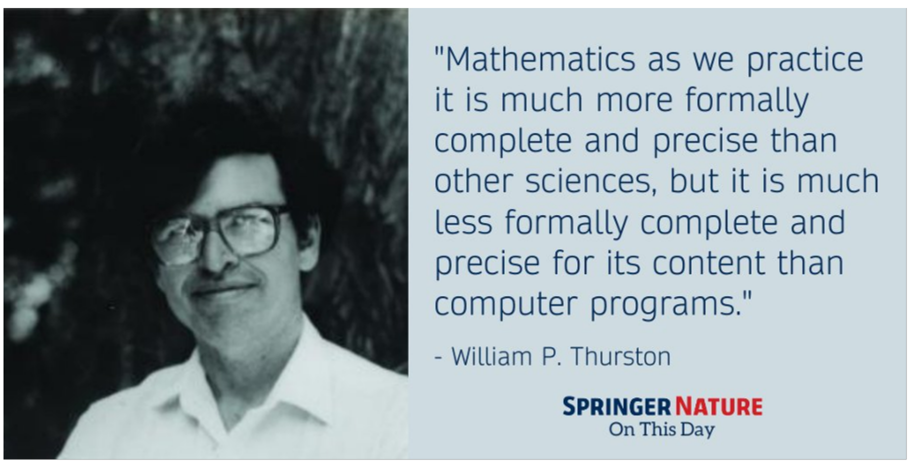


FIGURE 3. William Thurston

three-space, first investigated by Henri Poincaré and later studied by Lars Ahlfors. Deformations of these groups were studied by Thurston, and further advances in quasi-conformal maps resulted because of his effort. Renowned for being an enthusiast for an unusual style of mathematical writing that was strong on intuition and short on proofs his publications most famously included 'The Geometry and Topology of 3-Manifolds (1979)' and 'Three-Dimensional Geometry and Topology (1997)'. Thurston died on August 21st, 2012 in Rochester, New York, of a sinus mucosal melanoma that was diagnosed in 2011 which he bravely battled. He is remembered in Princeton for generous distribution of his time of ideas. For instance, in his period in Princeton he had 29 PhD students and is famous for his deep intrigue about the process of doing mathematics and methods for communicating it. [4]

3.3 **Awards.**

- Fields Medal (1982)
- Oswald Veblen Prize in Geometry (1976)

- Alan T. Waterman Award (1979)
- National Academy of Sciences (1983
- Leroy P. Steele Prize (2012).

4 Stephen Smale.

4.1 **Background.** Stephen Smale was born in Flint, Michigan in 1930. He was raised on a farmhouse and attended an elementary school with just one classroom. After years of somewhat mediocre grades in chemistry, physics, and nuclear physics, Smale focused on his studies in mathematics and was awarded a BS in 1952 from the University of Michigan. Smale started to excel in the field and was awarded a Ph.D in 1957 for the thesis Regular Curves on Riemannian Manifolds. In his thesis he generalised results proved by Whitney in 1937 for curves in the plane to curves on an n-dimensional manifold. Smale later began to apply topological methods to study problems relating to Pontryagin's work on structurally stable vector fields. Smale has worked as a professor at the University of California, Princeton, and the City University of Hong Kong. [5]
[13]

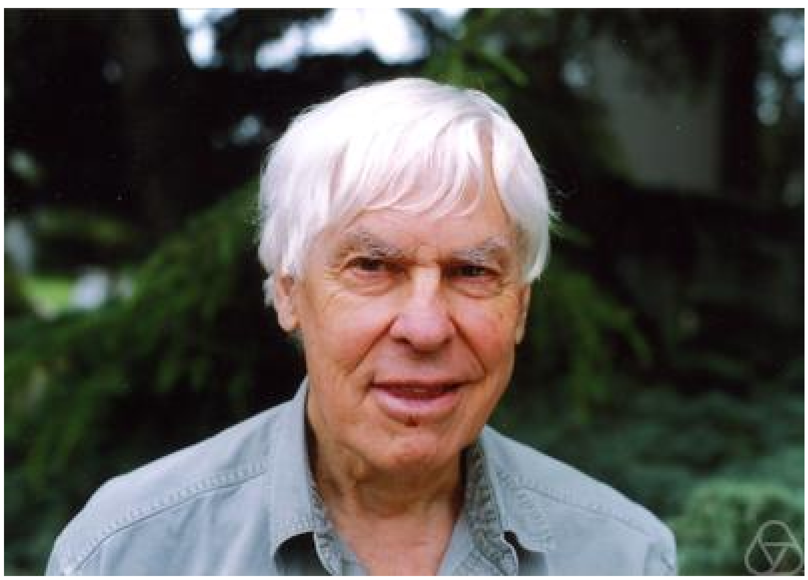


Figure 4. Stephen Smale

4.2 **Fields Medal.** Smale was awarded a Fields Medal at the International Congress at Moscow in 1966. Smale earned this accolade for his contributions to Topology, in particular, his work on the generalised Poincare Conjecture.
The Poincaré conjecture asserts that a simply connected closed 3-dimensional manifold is a 3-dimensional sphere. The higher dimensional Poincaré conjecture claims that any closed n-dimensional manifold which is homotopy equivalent to the n-sphere must be the n-sphere. When $n = 3$ this is equivalent to the Poincaré conjecture. Smale proved the higher dimensional Poincaré conjecture in 1961 for n at least 5. [5]

Smale's acceptance of the award was shrouded in controversy. Smale, a well-known political activist, had protested against the Vietnam War. To accept the award he had to travel to Moscow in the midst of the Cold War. Smale had been served with a subpoena at the time his award was announced. When he flew to Moscow to accept the award, many felt Smale was seeking refuge in the USSR. Despite pressure from president-to-be Ronald Reagan for Smale to be fired, Smale remained at Berkeley for three decades before moving to Hong Kong. [15]

4.3 **Later Work.** Smale also made significant strides in dynamical systems. His first contribution is the Smale horseshoe that started significant research in dynamical systems. In 1998 Smale compiled a list of mathematical problems to be solved in the 21st century, dubbing the list Smale's Problems. While many of these problems have been partially resolved, only three have been fully resolved. Since 2002, Smale has worked as a Professor at the Toyota Technological Institute at Chicago. Smale also received the Wolf Prize in 2007 for his contributions to mathematics.[13]

5 Vaughan Jones.

5.1 **Background.** Sir Vaughan Jones was born in Gisborne, New Zealand in 1952. He completed his undergraduate degrees at the University of Auckland, after studying mathematics and physics. He was awarded a scholarship in Switzerland to complete his PhD at the University of Geneva in 1979. His thesis, titled "Actions of finite groups on the hyperfinite II factor", won him the Vacheron Constantin Prize.[1]

In the 1980s he worked at the University of California, Los Angeles, and the University of Pennsylvania, before being appointed Professor of Mathematics at the University of California, Berkeley. Up until his death at the end of last year, taught at Vanderbilt University as Stevenson Distinguished Professor of mathematics. He remained Professor Emeritus at University of California and was a Distinguished Alumni Professor at the University of Auckland. The Jones Medal, created by the Royal Society of New Zealand in 2010, is named after him.[14]

5.2 **Fields Medal.** In 1990, Jones became the first person from New Zealand to be a recipient of the Fields Medal, for an idea that revolutionised the field of topology. Upon receiving the medal, he famously wore the New Zealand rugby jersey when he gave his acceptance speech at the International Congress of Mathematicians in Kyoto. His discovery, which is now called the Jones polynomial, was discovered while he was working on a problem in Von Neumann algebra theory where he discovered an unexpected link between that theory and knot theory. This breakthrough created a new mathematical expression for distinguishing between different types of knots as well as links in 3-dimensional space.[8]. It was a powerful way of telling knots apart as it could distinguish most knots from their mirror images. Jones discovery had been missed by topologists for the previous sixty years.

Since its discovery, the Jones polynomial has assisted in solving long-standing problems outside of topology, as researchers have found a lot of connections between knot theory and physics. Theoretical physicists have shown that the polynomial can be explained as a property of the quantum physics of the universe with simplified laws of nature and just two dimensions of space.[2] One application is

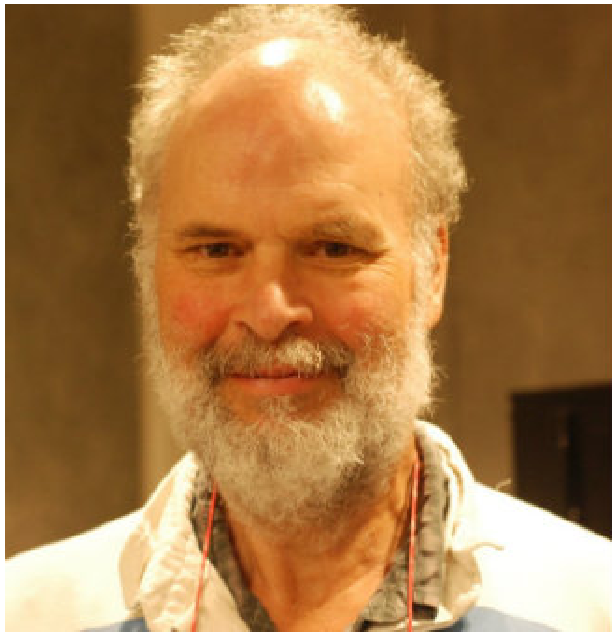


FIGURE 5. Vaughan Jones

where electrons in certain ultracold devices are confined to one layer, collective quantum states called anyons can be formed. These anyons are seen as a possible platform for building future quantum computers. Essentially, by calculating Jones polynomials, quantum algorithms can be run. Jones' work on knots has also been of interest in other areas such as statistical mechanics, functional analysis and molecular biology where it has been used to look into the knotting of DNA molecules.

5.3 **Later Work and Achievements.** Jones was a fellow of the Royal Society in both the United Kingdom and New Zealand and a Knight Companion of the New Zealand Order of Merit. He was also involved in many professional organizations of his peers, serving as vice president of both the American Mathematical Society and the International Mathematics Union.[8] In 2002 he was elected as an honorary Member of the London Mathematical Society and awarded an honorary degree by the Universite du Littoral, Cote d'Opale.[6] Jones was notably involved in the finding of the best way to characterize the group-like object arising from the tower of factors, which led him to a study of the Thompson groups and again to unexpected spin-offs for the theory of knots and links.

Jones' main focus continued to be Von Neumann algebras and their connections to other fields. He made many other contributions to the mathematical community in his editorial work for journals such as the Transactions of the American Mathematical Society and Reviews in Mathematical Physics.

References

1.
2. Davide Castelvecchi, *Vaughan jones (1952-2020).*

3. The editors of Encyclopedia Britannica, *Sergei novikov russian mathematician.*
4. ———, *William paul thurston american mathematician.*
5. MacTutor, *Stephen smale.*
6. ———, *Vaughan frederick randal jones.*
7. John Milnor and James D. Stasheff, *Characteristic classes*, Annals of Mathematical Studies, Princeton University Press, 1976.
8. Ann Marie Deer Owens, *Vaughan jones, preeminent vanderbilt mathematician, has died.*
9. International Mathematical Union, *Fields medals 2014.*
10. ———, *Fields medals 2018.*
11. unknown, *Fields medal.*
12. ———, *Segei novikov (mathematician).*
13. ———, *Stephen smale.*
14. ———, *Vaughan jones.*
15. Wendy Zukerman, *The extraordinary winners of the fields medal for mathematics.*

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

KNOT THEORY

GRAINNE GALLINAGH, ORAN O'MALLEY, AND JACK FLOOD

ABSTRACT. The following research paper was created to shed light on the fundamentals and history of Knot Theory. This paper will primarily focus on the explanation of knots, main goals of knot theory and the origins of knot theory.

1. Introduction: What is a Knot?

Knot Theory, a lively exposition of the mathematics of knotting. Knot theory is the study of mathematical knots and their properties. A knot is a three-dimensional closed curve which is homeoomorphic to a circle existing in the Euclidean space, R3. In layman's terms, to create a knot you would take a string, tangle it, and join the two ends together to form a continuous loop (The knot does not intersect itself at any point). This differs from a knot we would see in everyday life such as tying your shoelace or tying a knot in a rope as these are not connected at both ends. Another difference between a mathematical knot and a physical one is that mathematical knot has not thickness. It is a line which has a cross-section consisting of a single point. Once the knot is made it cannot be undone without breaking the loop. (Adams, 2000). However, some closed curves can be untangled i.e., it can be deformed into an unknotted curve. This unknotted curve is the simplest form of a knot which is known as an 'unknot' and resembles a circle. An important question is whether two closed curves, K1 and K2, are different or "are they the same knot in the sense that one can be continuously deformed into the other?". (R. Osserman) This may be difficult to visualise but as shown below [1], one knot can have different deformations. There are various types of knots which we will be discussing shortly.

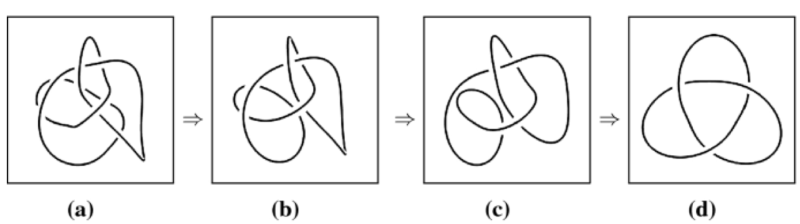
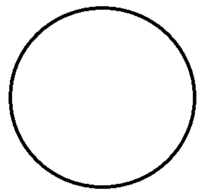


Figure 1. One knot – different deformations

2020 *Mathematics Subject Classification.* Primary .

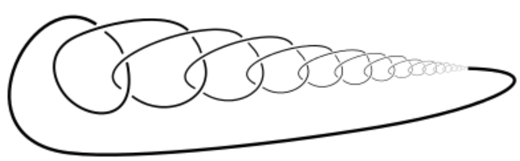
2. Types of Knots

2.1. **Trivial Knot.** The unknot, often known as the trivial knot, is a simple round circle embedded in R3. The trivial knot consists of zero crossings.

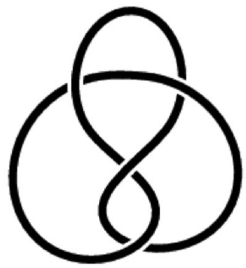


2.2. **Trefoil Knot.** Moving from trivial to untrivial we have the simplest untrivial knot called the Trefoil Knot. The Trefoil Knot is the only knot with crossing number 3, meaning it consists of 3 overlapping crossings.

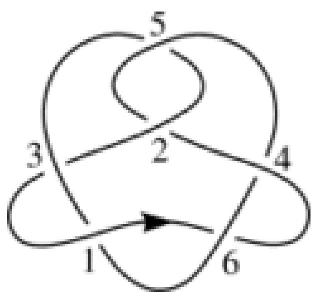
2.3. **Tame and Wild Knots.** A tame knot is any knot equivalent to a polygonal knot. A polygonal knot is a knot whose image in R3 is the union of a finite set of line segments. Knots which are not tame are called wild. Wild knots can have irregular and deviant behavior. The picture shown below is an example of a wild knot. Notice how the stitches and arcs get smaller and converge to a point? That point is called a wild point.



2.4. **Figure of 8 Knot.** A figure-eight knot (also called Listing's knot) is the unique knot with a crossing number of four. This makes it the knot with the third-smallest possible crossing number, after the unknot and the trefoil knot. It is a 2-embeddable knot, and is amphichiral (A knot that is capable of being continuously deformed into its own mirror image) as well as invertible (A knot that can be deformed via an ambient isotopy into itself but with the orientation reversed).



2.5. **Stevedore's Knot.** The stevedore knot is one of three prime knots with crossing number six, the others being the 62 knot and the 63 knot.The mathematical version of the knot can be obtained from the common version by joining together the two loose ends of the rope, forming a knotted loop. It can also be described as a twist knot with four twists, or as the pretzel knot. The Stevedore knot is invertible but not amphichiral.



3. Goals and Applications of knot theory

Essentially, the main goal of knot theory is to distinguish between two or more knots [1], but that completely undermines the implications that knot theory has in the world around us. Knot theory is a very theoretical subject; therefore its more practical ramifications may be lost on those who do not study it. However, chemists and physicists like William Thomson, 1st Baron Kelvin, who (incorrectly) theorized that atoms of different elements were distinguished by various knots in each [2] were the ones who created knot theory. There are applications for knot theory in molecular biology, cryptography, string theory and many more.

3.1. **Biology.** In one of the most astonishing discoveries of the twentieth century, F.H.C. Crick and J.D. Watson discovered the basic structure of DNA. They were jointly awarded the Nobel Prize for Medicine in 1962 for their insight into the nature of biological matter. Topology, specifically knot theory, aids in the unravelling of DNA knots. DNA is represented by two lengthy strands that have been entwined millions of times, knotted, and wrapped excessively. Researchers may precisely arrange the data and make it easier to grasp when they use knot theory to uncover the mysteries stored inside the strands of DNA.

3.2. **Chemistry.** Different qualities can be found in molecules made up of identical atoms. Molecular diagrams are used to determine the origin of these variations. These molecules were discovered to form knots and links. Knot theory could now be used by chemists to explain the different functions.

3.3. **Cyptography.** Programming, notably encryption, is another use of knot theory. Cyber security is an ever-present issue in an increasingly computerized environment. Encryption is essential for keeping your private messages confidential. A form of knot-based encryption system is being developed. It would work in a similar way as the RSA. A B share a finite list of prime knots. The message Is built up from a finite sequence of knots. Through a standard RSA protocol, B sends to A an ordered sub list of N prime knots. These composite knots are now sent to B. At this stage everyone has access to these strings of relative integers. B receives the composite knots. Since B knows the relevant prime knots, they can decrypt the composite knots, and thus decrypt the message

4. Origins of Knot Theory

The idea of knots has been around for centuries. Humans have been using them throughout history in many ways. Sailors use knots to measure the speed of their boats or ships. They also use them to secure objects to the deck of the boat. They were also widely used for decoration such as in 'The Book Kells' which contains many Celtic knots. Many mathematicians have tried to take the concept of knots and view them as a mathematical entity; however, it wasn't until 19th century that Carl Fredrich Gauss made progress with knot theory. One of the oldest notes found among Gauss' possessions was a set of knot drawings dated 1794 (Colberg, 2017). He concluded that if he labelled the crossings with letters, A, B, C etc., and noted the sequence of letters from a starting point in the knot until they reached the beginning again. He came to the conclusion that a knot with n crossings has a sequence of 2n letters. This was called "the scheme of the knot" (Colberg, 2017). It was not until after this, through his study of electrodynamics, that he developed 'The Gauss Linking Integral'. This was used for calculating the linking number of two separate knots. It represents the number of times each curve winds around the other. This number is unchanging under ambient isotopy as it does not change under smooth deformation of the loops. Gauss' work inspired many other mathematicians working in the field of topology to study knots and advance knot theory even further.

John Benedict Listing (1808 – 1882) was a doctoral student of Gauss who later coined the word "topology". He combined the two Greek words topos (form) and logos (reason). (He is also responsible for numerous other popularly used scientific terms and phrases, such as "nodal points," "telescopic system," and "micron," to name a few.) He went on to publish the article "Vorstudien zur Topologie" with large sections of it focusing on knots. His focus was knots and their mirror image. He discovered left and right trefoil knots are not equivalent, and showed that the figure eight knot and its mirror image are equivalent (Przytycki, 1992).

The real breakthrough occurred later in the century, when three Scottish mathematicians worked to create the first knot tables. Their names were Peter Guthrie Tait, Thomas Kirkman and Charles Newton little. Although Tait is credited with the first major contribution to the classification of knots with his work beginning around 1867, it was mathematician Thomas Kirkman who made the first substantial contribution to the classification of knots. (Colberg, 2017). For a long time, Kirkman's primary topic of study was this tabulation for alternating knots. He had alternate knots with up to eleven crossings on his tables. Then Tait teamed up with Little to continue the project, and the two experimented with notations offered by the likes of Gauss and Listing. They eventually decided on a better listing notation that was less ambiguous. Little and Tait went on to make further key discoveries in the area of knots as a result of their use of these tables.

5. Bilbliography

Adams, C. C., 2000. The Knot Book An Elementary Introduction to the Mathematical Theory of Knots. 1 ed. s.l.:W.H Freeman ans Company.

Colberg, E., 2017. A brief history of knot theory. P´agina consultada a, 8.

Osserman, Robert. "Knot theory". Encyclopedia Britannica, https://www.britannica.com/science/knot-theory. Accessed 2 June 2021.

Przytycki, J.H., 1992, Knot Theory from Vandermonde to Jones, Pg. 173-185, Aportaciones Matematicas Comunicaciones 11.

Weisstein, Eric W. "Figure Eight Knot." From MathWorld–A Wolfram Web Resource

FIELDS MEDAL AND SOME OF IT'S EARLY RECIPIENTS FOR WORK IN TOPOLOGY

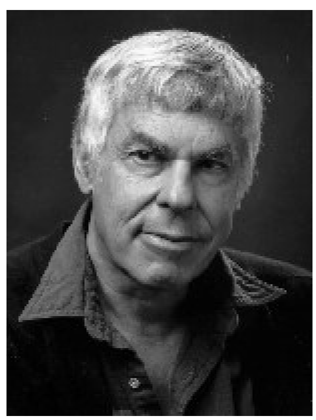
PAVOL HARSANIK AND BRENDAN O'DONOGHUE

This paper is dedicated to our advisor Graham Ellis.

Abstract. This paper is a short view into Fields medal and some of its' earliest recipients for their work in a field of TOPOLOGY. The Fields Medal is a prize awarded to two, three, or four mathematicians under 40 years of age at the International Congress of the International Mathematical Union (IMU), a meeting that takes place every four years.

The Fields Medal is regarded as one of the highest honors a mathematician can receive, and has been described as the mathematician's Nobel Prize. The prize is named after Canadian mathematician John Charles Field. Unlike Nobel's Prize, it is awarded every four years. It is considered to be, one of the highest honours a mathematician can recieve.

1. Stephen Smale



History and Background

(born July 15, 1930 in Flint, Michigan) is an American mathematician, who is known among other achievments, for his early work in topology. In the young age, his favorite subject was Chemistry, and later on he switched his interest into

Date: June 1st, 2021.
2020 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.
Key words and phrases. Fields Medal, Topology.

physics. After failing the physics course, he turned to mathematics, and that's where he found his glory.

Smale was known for his political activism and in 1965, he took 6 months break from mathematical research, to the first campaign of nonviolent civil disobedience directed at ending U.S. involvement in the Vietnam War.

The following year(1966), he received the Fields Medal at the International Congress of Mathematicians in Moscow. During the press conference he controversially criticised both U.S. and Soviet governments. This happened in the middle of cold war and has sparked quite a controversy. During the conference itself, he has been abruptly taken for a car ride with soviet agents to have a 'talk', although he said he has been treated with courtesy.

He was awarded Fields medal for his work, based on iconic paper called "Finding a Horse shoe on the Beaches of Rio"

Fields Medal

In 1960, he achieved his two most famous mathematical results. First, he constructed a Horshoe, function that that serves as a paradigm for Chaos. Next, he proved generalized Poincaré conjecture, for all dimension greater or equal to five. Two-dimensional version has been established in 19th century and three-dimensional early in 21st century. Smale's work was brilliant because of bypassing 3-d and 4-d to resolve the problem for all higher dimensions. He continued his work, and in 1961 he came up with the h-cobordism theorem, which became the fundamental tool for classifying different manifolds in higher-dimensional topology. It is considered one of the most important theorems in topology.

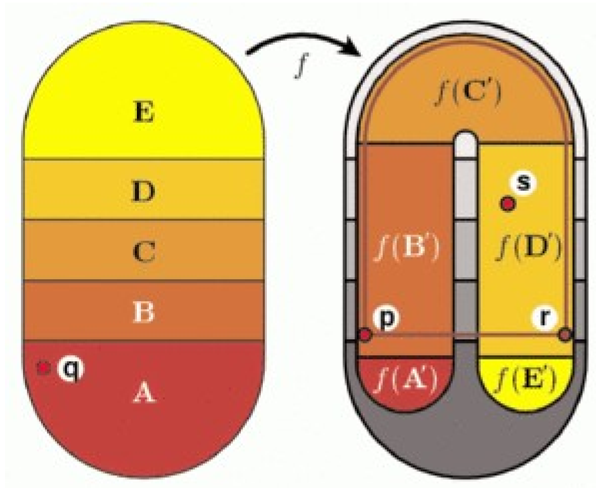
As Smale mentioned in his paper, he was driven by unsolved problem presented by Poincare. Since the start of his research in mathematics, he produced false proofs of the 3-dimensional Poincare Conjecture. He was returning to this again and again. Within two months of finding the horseshoe, on the beaches of Rio, he came with the idea which brought him back to Poincare's assertion, and he limited it to dimensions 5 and higher.

The rough idea of his work, is represented in embedding function of a disc, to itself as shown on the diagram.

It contracts the semi-discs A , E to the semi-discs f(A) , f(E) in A ; and it sends the rectangles B , D linearly to the rectangles f(B) , f(D) , stretching them vertically and shrinking them horizontally. In the case of D , it also rotates by 180 degrees.

There are three fixed points p,q and s and a homoclinic point - r. Point q is the sink,all points z $\in [A \cup E \cup C]$ converge to q under forward iteration, fn(z)$\rightarrow$q as n$\rightarrow\infty$.

The points p , s are saddle points. If x lies on the horizontal through p then fn squeezes it to p as n$\rightarrow\infty$,while if y lies on the vertical through p then the inverse iterates of f squeeze it to p . With respect to linear coordinates centered at p, f(x,y) = (kx,my) where (x,y) $\in [B]$. Similarly, f(x,y) = (kx,my) with respect to linear coordinates on D at s .



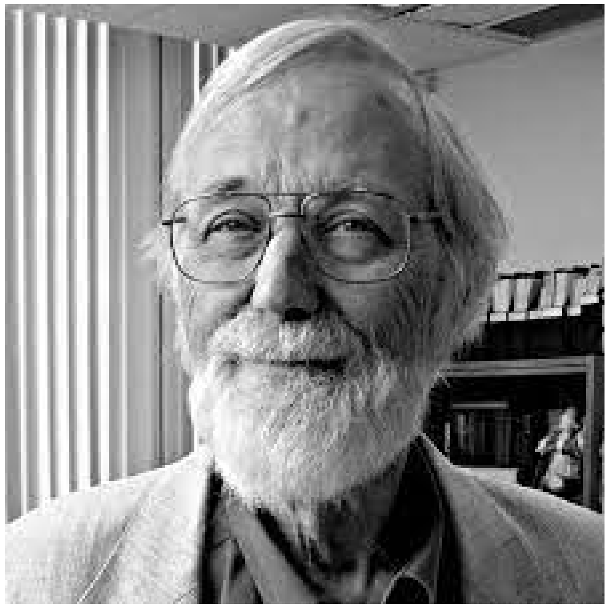
The sets
Ws=z:fn(z)→p as n→+∞,
Wu=z:fn(z)→p as n→∞,
are the stable and unstable manifolds of p . The homoclinic point r here is transverse in the sense that the stable and unstable manifolds are not tangent at r . The figure only shows these invariant manifolds locally. Iteration extends them globally.

The utility of Smale's analysis is, that every dynamical system having a transverse homoclinic point, such as r , is such that some power fT has also a horseshoe containing r , and has thus the shift chaos.

The idea produced not only only solution to Poincare's conjecture in dimensions greater then 4, but it also gave rise to many amazing discoveries in topology, and other fields of mathematics.

.

2. John Milnor



History and Background

John Milnor was born in New Jersey the 20th of February 1931. His mathematical work began in 1951 in Princeton University where he wrote a thesis on Link Groups [4]. While he was still an undergraduate at Princeton University, he was one of two people who proved a theorem which stated that slightly curved surface could not be knotted [6].

In 1954, he then went on to receive a PHD in mathematics where he continued his links studies on the Isotopy of links. After receiving his PHD, he continued his work at Princeton University right up until 1967. During that time he won his field medal in 1962 for his work on a new strand of differential topology. He proved that a 7 dimensional sphere can have at least 7 differentiable structures. His rationale for studying this area was to try and bring more geometry into topology [2].

He received other awards in mathematics such as the Wolf Prize in 1989 for his work on his further findings on geometry which changed the lens of topology. He also won the Abel prize in 2011 for his work on smooth spheres [8].

Fields Medal

The 'exotic sphere' was a unique type of sphere discovered by Milnor. This particular sphere was a manifold that was homeomorphic and not diffeomorphic to S^7 (7-dimensional sphere). A diffeomorphism is similar to a homeomorphism except it only involves the bijective mapping of smooth differentiable manifolds [1].

One of Milnor's first contributions towards exotic spheres was his discovery that there are some S^3 bundles over S^4 with the rotation group $SO(4)$ that are homeomorphic and not diffeomorphic [7]. A bundle or fibre bundle. Eells and Kuiper then found that 15 out of the 27 differentiable manifolds can be expressed in the form of S^3 bundles over S^4. This form of bundle is called a fibre bundle [5].

A fibre bundle defined in Nicholson's article is two spaces F and X in a space E where a function $f : E$ exists such that there is an open cover U_i of X and homeomorphisms $h(i) : f^{-1}U_i \to U_iF$ corresponding with f in the first coordinate:

$$F \to E \to X \tag{2.1}$$

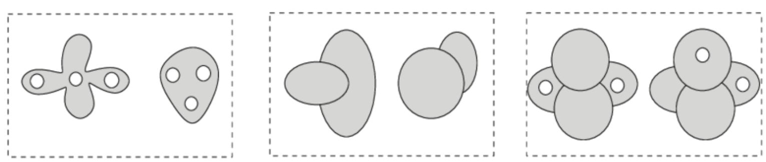
In other words, this means that each fibre $f^-1(x)$ forms a homeomorphism with xF where E is the total space, X is the Base space and F is the fibre [9].

Lastly, another piece of important work which contributed to Milnor's discovery of exotic spheres was his work on the λ invariant. The λ invariant is a homotopy seven sphere denoted as Σ^7. Σ^7 then is the boundary of any compact manifold W8 which forms a linear combination in $\mathbb{Z}/7$ of the Pontryagin number $p_1^2[W^8]$[10].

$$\lambda(\Sigma^7) = 2p_1^2[W8] - \sigma(W8)mod7 \tag{2.2}$$

Where the Pontryagin number is defined in terms of the Pontryagin class which are certain characteristics of real vector bundles. These type classes have a degree of a multiple of 4 [3].

Figure 2 below shows an example of spaces that are homeomorphic but not diffeomorphic, diffeomorphic and lastly, not homeomorphic.



homeomorphic but not diffeomorphic diffeomorphic not homeomorphic

Based on first diagram in figure 2, it is evident that there are points in the topological space that are not differentiable (i.e. does not have a tangent).

References

1. Smale, S., 1998. Finding a Horseshoe on the Beaches of Rio. [online] Cityu.edu.hk. Available at: ¡http://www.cityu.edu.hk/ma/doc/people/smales/pap107.pdf¿.
2. Maths History. 2021. Stephen Smale - Biography. [online] Available at: ¡https://mathshistory.st-andrews.ac.uk/Biographies/Smale/¿.
3. Smale,Shub, S., 2007. Smale horseshoe. [online] Scholarpedia. Available at: ¡http://www.scholarpedia.org/article/Smale$_h$*orseshoe* > *.M.AbateExotic Spheres and John Milnor. In Imagine Math 2Springer, Milano* **33**(2013), 221− −229.

4. Britannica *The Editors of Encyclopaedia. "John Willard Milnor". Encyclopedia Britannica, 16 Feb. 2021, https://www.britannica.com/biography/John-Willard-Milnor. Accessed 3 May 2021* Springer, Milano
6. *https://encyclopediaofmath.org/wiki/Pontryagin$_c$lass*
7. *http://www.math.toronto.edu/mein/teaching/MAT367/DiffGeomNotes.pdf*
8. J. Eells, N. Kuiper *An invariant for certain smooth manifolds* Annali di Math **33** (1962), 93–110.
9. J J O'Connor, E F Robertson *John Milnor biography* University of St Andrews Scotland **33** (2015).
10. J. Milnor *Classification of (n-1)-connected. Surveys on Surgery Theory: Papers Dedicated to CTC Wall*, 1, 25.
11. M.Raussen, J.Milnor *Interview with John Milnor* Notices Amer. Math. Soc. 59 (3) **33** (2012), 400–408.
12. J. Nicholson *Exotic Spheres*
13. D.J. Wrazidlo *Detecting Exotic Spheres Via Fold Maps* Institute of Mathematics for Industry, Kyushu University.

Department of Mathematics, NUI Galway
Current address: Department of Mathematics NUI Galway
Email address: xyz@math.university.edu

Department of Mathematics, NUI Galway
Email address: two@maths.univ.edu.au

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

AN EXPLANATION OF THE MAPPER CLUSTERING ALGORITHM DUE TO SINGH, MEMOLI, AND CARLSSON

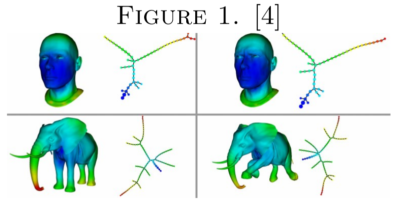
THOMAS HAYES AND CIAN DOHENY

ABSTRACT. The idea behind mapper is to take a data set and output a graph that represents the data set. The data Mapper algorithm is particularly useful in examining large complex data where the usual summary statistics don't offer much insights or are misleading. We will begin by highlighting the differences between this approach and existing approaches, a simple explanation with diagrams and a worked example using the R package.

1. INTRODUCTION

The Mapper Algorithm was introduced in 2007 in the paper "Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition" [1]. Authored by Singh, Mémoli, and Carlsson. The idea behind mapper is to take a data set and output a graph that represents the data set. The graph can reveal hidden structures within the data.

FIGURE 1. [4]

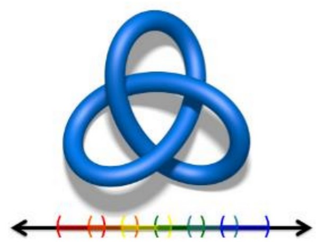


2. DIFFERENCES FROM EXISTING APPROACHES

The Mapper algorithm takes a novel topological approach in looking at the data. Existing approaches tended to be more sensitive on the metric. The simplicial complex output is combinatorial and therefore not required to be embedded in Euclidean space. There are some approaches that also produce combinatorial output such as disconnectivity graphs and cluster trees. However they are all tree graphs that are only one dimensional. The topological approach used is more robust and can be used with a much wider range of data s [3].

2010 *Mathematics Subject Classification.* Primary .

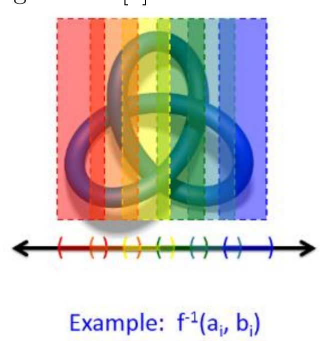
FIGURE 2. An example of a high dimensional data set which *Mapper* would reduce to simplical complexes [2].



3. How does the algorithm work

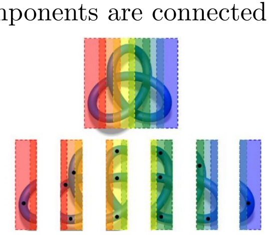
3.1. **Filtering function.** The goal of *Mapper* is to take a high dimensional data set and be able to reduce it into simplicial complexes. The first step is applying a 'filtering function' to the high dimensional data sets in order for us to be able to make some clear distinctions between data points. The function maps our data set to the Real line separating the data into overlapping intervals.

FIGURE 3. This is an example where *Mapper* maps the data set to the Real line using a filter [2].



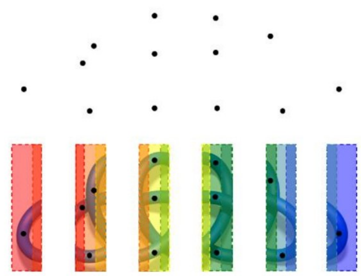
3.2. **Pre-image.** Using the pre-image *Mapper* will take some open intervals along the Real line and put the data found in the pre-image within these intervals into separate bins from here vertices are chosen using a clustering algorithm.

FIGURE 4. Here *Mapper* will look at the pre-image and determine vertices and what components are connected [2].



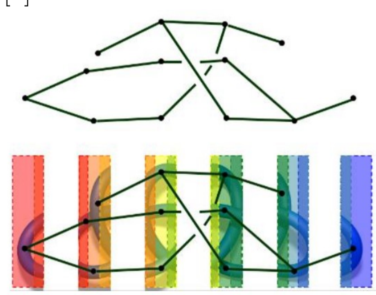
3.3. **Overlapping Intervals.** Given the new representation of data we will now further examine the bins that were just made. *Mapper* takes these bins which are divided by some open interval on the real line. The chosen open intervals were done so as to have some overlapping take place. The idea behind the overlapping bins is so to form a connected graph, we take remark of connected particles in adjacent.

FIGURE 5. As seen in this diagram there are separate bins with vertices and the overlapping intervals will help determine the connected particles [2].



3.4. **Connected components.** The algorithm looks for overlapping clusters between pre-images. IF there is an overlappping clusters (which are now vertices) then an edge is drawn between them to form connected components.

FIGURE 6. Mapper has now drawn a map from all connected vertices which has given us a resulting simplicial complexes for the original data set [2].



4. AN EXAMPLE USING TDAMAPPER

Here we will present an example using using TDAMApper package in RStudio. Data was generated in a spiral shape in $\mathbb{R}^2$(Figure 6).

4.0.1. *Example.* Here the data is projected onto the y axis and divided into 10 overlapping bins with a 50 percent overlap. Another example of a filtering function would be projecting data points in $\mathbb{R}^3$ to $\mathbb{R}$ f(x,y,z) -¿ f(x) and dividing it into n equal overlapping bins.

Figure 7. 2-d Spiral of data points

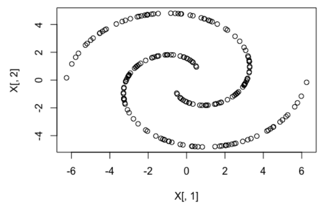
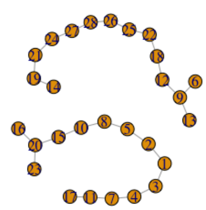


Figure 8. Rcode used

```r
filter <- X[,2] # height projection
num_intervals <- 10
percent_overlap <- 50
num_bins_when_clustering <- 10

m3 <- mapper1D(
  distance_matrix = d,
  filter_values = filter,
  # num_intervals = 10, # use default
  # percent_overlap = 50, # use default
  # num_bins_when_clustering = 10 # use default
)
```

Figure 9. Simplicial complex generated



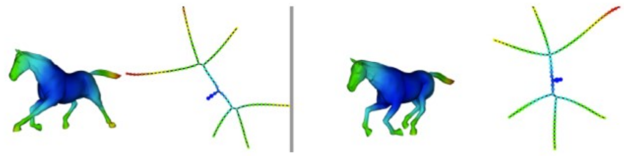
5. Applications

An interesting application of the mapper Algorithm is using it alongside machine learning to recgonise 3D objects. A point cloud data set is used as input. The simplicial complex generated by the algorithm is much easier to work with and reduces the point cloud data down to the most important information. A simplicial complex generated for a point cloud data of a horse can be seen below.

Another simplicial complex is generated from a horse in a different position. This horse is topologically the same as the first horse. The simplicial complex generated for the second horse is very similar to the first horse because of the topological approach used in the mapper algorithm.

This is extremely useful when used in combination with machine learning. The simplicial complex is an easy dataset to use as input and lead to improved accuracy with less training data.

FIGURE 10. [2]



REFERENCES

1. *Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition from* Singh, G., M'emoli, F., Carlsson, G. E. (2007), https://diglib.eg.org/bitstream/handle/10.2312/SPBG.SPBG07.091-100/091-100.pdf?sequence=1isAllowed=y
2. *An Introduction to a New Text Classification and Visualization for Natural Language Processing Using Topological Data Analysis from* Naiereh Elyasi, and Mehdi Hosseini Moghadam ,https://arxiv.org/ftp/arxiv/papers/1906/1906.01726.pdf
3. *The Mapper algorithm and its applications* from Boris Goldfarb, University at Albany, SUNY, http://topology.nipissingu.ca/workshop2018/slides/Goldfarb-Data-Beamer.pdf
4. *Ohio State University* https://research.math.osu.edu/tgda/mapperPBG.pdf

Email address: t.hayes5@nuigalway.ie

Email address: c.doheny2@nuigalway.ie

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

AN INTRODUCTION TO ALGEBRAIC TOPOLOGY AND USING BETTI NUMBERS TO DISTINGUISH BETWEEN TOPOLOGICAL SPACES

ANTHONY HORGAN

(Communicated by Professor Robert K Lazarsfeld)

Abstract. This paper is meant as an introduction to algebraic topology. Specifically it focuses on Betti numbers, what they are and how they are calculated. Concepts from algebraic topology will be introduced and discussed in order to better understand Betti numbers.

1. Introduction

Algebraic topology uses concepts and tools from abstract algebra to study topological spaces. Betti numbers are an important concept in algebraic topology and have uses in topological data analysis (analysing datasets using techniques from topology). Betti numbers are topological invariants which can be used to distinguish between topological spaces based on the connectivity of their corresponding simplicial complexes. Intuitively the d^{th} Betti number of a topological space can be thought of as the number of d-dimensional holes it has.

Sources for this report include [1]

2. Simplicial Chains

Simplicial complexes can be broken down into their skeletons which contain only simplices of a certain dimension.

The p^{th} chain group of C_p of a simplicial complex K consists of all combinations of its p-simplices. For the purposes of convenience of implementation we will the coefficients to be $\mathbb{Z}_2$. That is, all elements of the p^{th} chain group are of the form $\sum_j \sigma_j$ for $\sigma_j \in K$. The group operation would be addition of elements σ_j with coefficients in $\mathbb{Z}_2$. (Note that the coefficients being elements of $\mathbb{Z}_2$ is not always the case and other choices for coefficients are also possible.)

The chain group of a simplicial complex are used to represent algebraically represent the concept of a boundary.

2020 *Mathematics Subject Classification.* Primary: 55-01; Secondary: 55N31.

3. The Boundary Homomorphism

The p^{th} boundary homomorphism of a simplicial complex K is a function that sends each simplicial complex $\sigma = \{v_0, v_1, ..., v_p\}$ to its boundary:

$$\partial_p \sigma = \sum_i \{v_0, ...\hat{v_i}, ..., v_p\}$$

where $\hat{v_i}$ indicates that v_i is not included. This function is a homomorphism between the chain groups. The boundary homomorphism takes all combinations of elements of K which make up its p^{th} boundary.

4. Cycle Groups and Boundary Groups

We define the cycle group of the boundary homomorphism to be its kernel:

$$Z_p = ker\partial_p$$

We define the boundary group of the boundary homomorphism to be its image:

$$B_p = im\partial_{p+1}$$

5. Homology Groups and Calculating Betti Numbers

The p^{th} homology group H_p is a quotient group defined by removing cycles that are boundaries from higher dimensions

$$H_p = Z_p/B_p = ker\partial_p/im\partial_{p+1}$$

We can now calculate the p^{th} Betti number:

$$B_p = rank(H_p)$$

The rank of a group being the cardinality of the smallest generating set for that group. Intuitively, this process can be thought of as counting all boundaries of a simplicial complex while taking away all boundaries that come from a higher dimension.

References

1. Mark Anthony Armstrong, *Basic topology*, Springer Science & Business Media, 2013.

National University of Ireland Galway
Email address: a.horgan5@nuigalway.ie

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

INTRODUCTION TO KNOT THEORY

THOMAS JACKSON AND AIBHILIN CRANGLE

(Communicated by Laura Turner)

ABSTRACT. The branch of mathematics known as topology can be used to help us understand a variety of other, seemingly unrelated, concepts. One of these is the study of knots, otherwise known as knot theory. While inspired by knots which appear in daily life, such as those in shoelaces and rope, a mathematical knot differs in that the ends are joined together so that it cannot be undone. In mathematical language, a knot is an embedding of a circle in 3-dimensional Euclidean space, $\mathbb{R}^3$. The challenge in knot theory is to distinguish when two knots are 'different' i.e when one knot cannot be manipulated into another knot. This can be done in a number of ways through the means of knot invariants, an example of which are knot polynomials. When discussing various knots, we can also see if 'adding' them leads to any new properties, or if we can form new knots from different ones.

1. History

Research in knot theory begins in 1833 by Carl Friedrich Gausse. His motivation was to find the linking number of two knots; that is, the number of times each string is wrapped or looped around the other.

After gaining some momentum, the research into knot theory accumulates. In 1867, William "Lord Kelvin" Thomson pondered whether atoms themselves were knots of swirling vortexes in the aether. Thomson worked alongside Scottish Physicist Peter Tate, as both were of the belief that a deeper understanding and classification of all possible knots was required in order to explain why atoms absorb and emit light at only discrete wavelengths. Thomson came to the conclusion that Sodium, due to its two lines of spectra, could be considered the Hopf link, the simplest non trivial knot consisting of two rings with a linking number of +1.

Tate began formulating what is known today as the Tate Conjectures, proved in the 1990s. He had listed types of knots as if it were a table of elements, later added to by scientists C.N. Little and Thomas Kirkman.

Meanwhile, James Clerk Maxwell develops his own study into Gauss' linking number, in the field of electromagnetic theory. He re-interpreted this number as the work done by a charged particle moving along a link (imagine: one of the string components of a knot) by the magnetic field generated by the electric current travelling along the other "string".

The afore-mentioned vortex theory was in turn discounted in the late 19th century, and the motivation to develop knot theory dissipated with it.

2. Knot equivalence

A knot is created by starting with a line segment, wrapping it around itself arbitrarily and then fusing the two ends together. In other words, it can be represented as a continuous and 'nearly' injective function

Received by the editors 2nd May 2021.
2020 *Mathematics Subject Classification.* Primary 57K10.

$$K : [0, 1] \mapsto \mathbb{R}^3 \tag{2.1}$$

where $K(0) = K(1)$. In topology, we consider knots to be equivalent if one can be pushed about smoothly, without cutting or self-intersecting, to create the other one.

A more formal mathematical definition is that two knots are equivalent if they are *homotopic* to one another. i.e. if there exists a continuous mapping

$$H : \mathbb{R}^3 \times [0, 1] \mapsto \mathbb{R}^3 \tag{2.2}$$

such that

- for each $t \in [0, 1]$, $H(x, t)$ is a homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$
- $H(x, 0) = K_1$, $H(x, 1) = K_2$

where K_1, K_2 are the respective knots.

3. Knot invariants

In the 1930's, Kurt Reidemeister came up with a series of moves to distinguish the "difference" between knots. All deformations can be reduced into three moves; the twist, the slde, and the poke. If you can transform a knot into another knot using any series of Redidermeister moves, the knots are equivalent. This leads to a trial and error process and it can become tedious to try to prove that two knots are the same. We develop what is called an invariant, which are characterstics of a knot which remain unchanged by Reidemeister.

A *knot invariant* is a 'quantity' that is the same for equivalent knots, similar to the Euler characteristic number for different topologies. Different knots may have the same values, so this method isn't completely reliable for differentiating knots. However if their values are different, then they are definitely different knots.

3.1. Tricolourability.

A knot's ability to be coloured in 3 different colours. There are some rules with this method:

- At least two colours must be used
- At each crossing the three incident strands are either all the same colour or all different colours.

When we compare the trefoil and the unknot, we can see that the trefoil can be coloured in 3 different colours and the unknot is just one colour. Because the unknot violates the first rule of tricoukability, we know for sure that these knots cannot be manipulated to be the same knot.

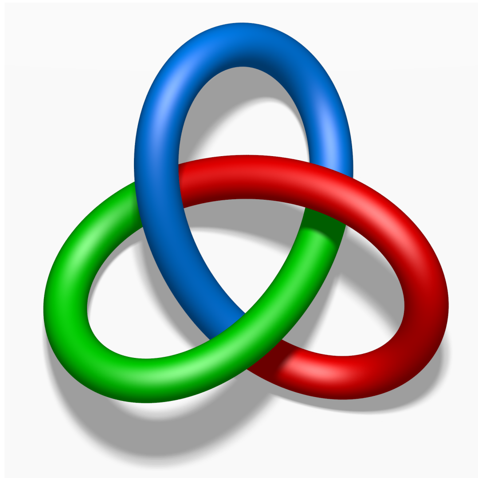


FIGURE 1. Tricoloring of the Trefoil Knot

3.2. **Knot Polynomials.** A knot polynomial is a knot invariant that assigns a polynomial to a certain knot, and whose coefficients of that polynomial relay insight to some properties of that knot.

The Alexander polynomial is computed by developing an (n, n+2) matrix based upon a knot with n crossings and n+2 regions. The matrix entries will only have the values 0, 1, -1, t, -t. Each of these values correspond to the position of the region at the crossing. If the region is on the left before undercrossing, the entry is -t, on the right before undercrossing, the entry is 1 and so forth.

After removing some columns, we can find the determinant of our new (n, n) matrix and this gives us the Alexander Polynomial.

Examples include the unknot with Alexander polynomial 1, the Trefoil knot with Alexander polynomial t - 1 + 1/t, and the Figure-eight knot with Alexander Polynomial -t + 3 - 1/t.

4. Adding knots

To add two knots, first cut both knots. Then join both knots at the cuts. The resulting knot is a sum of the original two knots. This is known as the *knot sum*. Depending on how the cuts are executed, no more than two new knots may be formed.

The knot sum is both *commutative* and *associative*, similar to the group of integers under addition. A knot can be considered *prime* if it is non-trivial, and cannot be written as the sum of two or more non-trivial knots. A knot that is not prime is considered *composite*. Similar to prime and composite numbers, a knot can be continuously and uniquely decomposed into one or more prime knots.

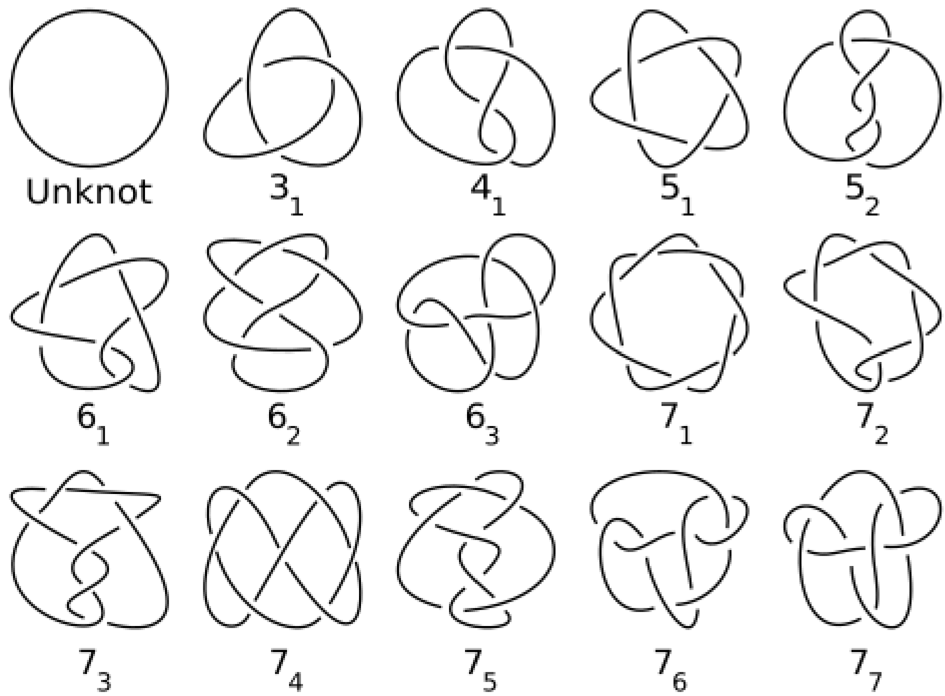


Figure 2. The first 15 prime knots

5. Conclusion

Knot Theory is a branch of topology with many different interesting aspects to it. From its extensive history, to attempting to differentiate between knots using various knot invariants, to creating new knots by adding them together, this has a variety of different applications, not only within the branch of mathematics itself, but also to the fields of biology, chemistry and physics.

6. References

What is a Knot? - Numberphile
https://www.youtube.com/watch?v=aqyyhhnGraw

Basic Topology - Armstrong

Wikipedia - Knot Theory
https://en.wikipedia.org/wiki/Knot_theory

Introduction to Alexander Polynomial
https://www.youtube.com/watch?v=b-qKIIBrWGg

Ballynalour, Saint Mullins, Co. Carlow, Ireland
Email address: T.Jackson1@nuigalway.ie

6 The Village, Wellpark, Galway, Co. Galway
Email address: A.Crangle@nuigalway.ie

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)000

Figure 1. Fields Medal

THE FIELDS MEDAL AND ITS RECIPIENTS.

KEYLEIGH MAGEE, ELLI-MAE MAGUIRE AND EMMA WATERS

Communicated by David Futer

Abstract. 1

In this paper we communicate the significance of a Fields Medal and its importance in Mathematics. We analyse the work of three topologists Vaughan Jones, Grigori Yakovlevich Perelman and Ren´e Fr´ed´eric Thom. All of which have been successful in obtaining a Fields Medal and whose contribution to mathematics has been particularly significant to the field of topology.

What is a fields medal?

The fields medal is regarded as one of the highest honours which a mathematician can receive and has an official title as International Medal for Outstanding Discoveries in Mathematics. The medal, which was named after the Canadian mathematician James Charles Fields is awarded to between two and mathematicians every four years at the International Congress of the International Mathematical union (IMU) to recognise and praise the achievements of the mathematician and their developing work. The fields medal is often referred to the 'Nobel Prize of Mathematics' as Alfred Nobel chose not to include mathematics in the areas of recognition the Fields Medal has became its substitute.

The first medal was awarded in 1936 and has now been awarded alongside a cash prize of 15,000 CA dollars which is funded by the J.C. Fields trust. The fields medal has recognised brilliance in a range of subjects of mathematics including mathematical physics, geometry, dynamic systems, differential equations and topology. The fields medal was last awarded in 2018 to Caucher Birkar, Alessio Figalli, Peter Scholze and Akshay Venkatesh and has been awarded to sixty mathematicians since its launch in 1932.

In this paper we will focus on topologists who have been awarded the Fields Medal.

Vaughan Jones

Vaughan Jones was awarded the Fields medal in 1990 aged 37 for his studies of functional analysis and knot theory. His studies have been pivotal in the area of topology as it has connected knots to quantum physics. Jones developed a new mathematical expression now known as the Jones polynomial that differentiates different kinds of knots specifically, it could distinguish most knots from their mirror images. The discovery triggered the development in a new are of mathematics called quantum topology that focuses on the three-dimensional spaces filled with holes and loops.

.

2. Career

Jones was born on December 31, 1952, Gisborne, New Zealand and studied maths and physics at undergraduate and masters level in the University of Auckland, New Zealand. He was successful in obtaining a scholarship for the University of Geneva, Switzerland where he achieved his doctorate in the École des Mathématiques alongside winning a F W W Rhodes Memorial Scholarship.

In 1980 Jones moved to the University of California, Los Angeles where he was an Assistant Professor of Mathematics. He later transferred to the University of Pennsylvania and was promoted to associate professor in 1984. In 1985 he was appointed as the Professor of Mathematics at the University of California, Berkley.

Vaughan Jones, has made many substantial contributions to the mathematical community and has been editor for numerous mathematical journals. In addition to the fields medal, Jones has been been awarded a Guggenheim Fellowship in 1986 and elected a Fellow of the Royal Society in 1990. In 1991 he was awarded the New Zealand Government Science Medal and given an honorary D.Sc. from the University of Auckland in 1992. In 2004 he was elected as Vice-President of the American Mathematical Society in 2004 Vaughan Jones died suddenly on Sept. 6th 2020 after complications following health complications.

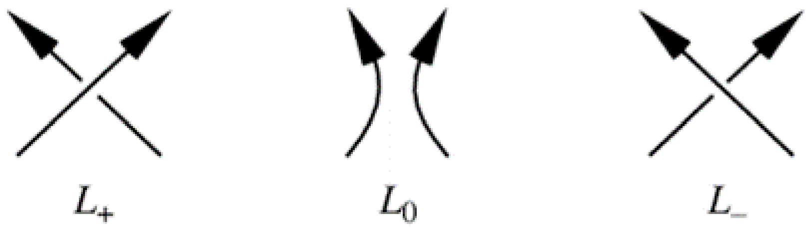


FIGURE 2. Jones Polynomial

2.1. **Jones Polynomial.** Jones polynomials are Laurent polynomials expressed in t assigned to a R3 knot. The Jones polynomials are denoted V L(t) for links and V K(t) for knots and follow 3 specific properties:

Lemma 2.1. *Let $f, g \in A(X)$ and let E, F be cozero sets in X.*

(1) *The Jones polynomial of any link (L) is contained in the ring*

$$\mathcal{Z}[t^1/2, t^-1/2] \tag{2.1}$$

(2) *The Jones polynomial on the un-knot K(O) = 1*
(3) *The Skein relation applies to the Jones polynomials and is used to evaluate either V(L+), V(L-) or V(L0) when two variables are known. - where the Skein relation is:*

$$t^-1V(L+) - tV(L) + (t^-1/2 - t^1/2)V(L0) = 0 \tag{2.2}$$

where L+, L, L0 are oriented links with diagrams that are equal except in a small region where they differ as follows above:
Where, L+ is a positive crossing, L- is a negative crossing and L0 is no crossing. It is the case that when any two links are crossed in one of these 3 ways and can also be manipulated from one type to another (Stephen Bigelow paper.)

Stephen Smale

Stephen Smale was born in Flint in Michigan. In 1966 he was awarded the Fields Medal for his research on topology in higher dimensions.

2.2. **Career.** He attended the University of Michigan to study physics but after failing changed to mathematics where he was awarded a BS and a MS. In 1957 he was awarded his PhD for the thesis of Regular Curves on Riemannian supervision. The following year Smale studied Pontryagin's work on structurally stable vector fields and how to employ topological methods to examine the these problems. He took on a fellowship at the Institute for Advanced Study at Princeton. He went on to became a professor of mathematics at the University of California at Berkeley and 1995 and took up a post as professor at the City University of Hong Kong. In 1966 Smale was awarded a Fields Medal at the International Congress at Moscow, for his work on the generalised Poincaré conjecture.

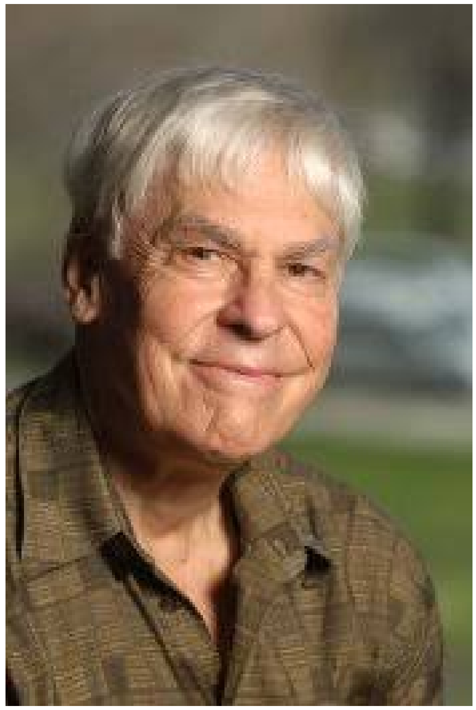


Figure 3. Stephen Smale

The Problem. The generalized Poincaré conjecture states that every homotopy sphere, which is a closed manifold of n dimension that is homotopy equivalent to the n-sphere, in one of three categories; topological manifolds, PL manifolds, or smooth manifolds, is homeomorphic to the n-sphere. The conjecture in dimension three seemed credible, while the generalized conjecture was considered to be false. Stephen Smale proved the Generalized Poincaré conjecture for dimensions greater than or equal to five. Smale's work was notable in that he looked past dimensions three and four to resolve the problem for all higher dimensions above four.

Rene Fre de ric Thom

Rene Fre de ric Thom was a French mathematician who, in 1958, at the International Congress of Mathematicians in Edinburgh, received the Fields medal for his work in covering the foundations of cobordism theory.

2.3. **Career.** Thom attended the now Universities of Paris in 1946. He went on to spent four years at the nearby National Centre for Scientific Research, and was awarded a PhD for his thesis in Sphere bundles and Steenrod squares. In 1964 he became a professor at the Institute of Advanced Scientific Studies, Bures-sur-Yvette.

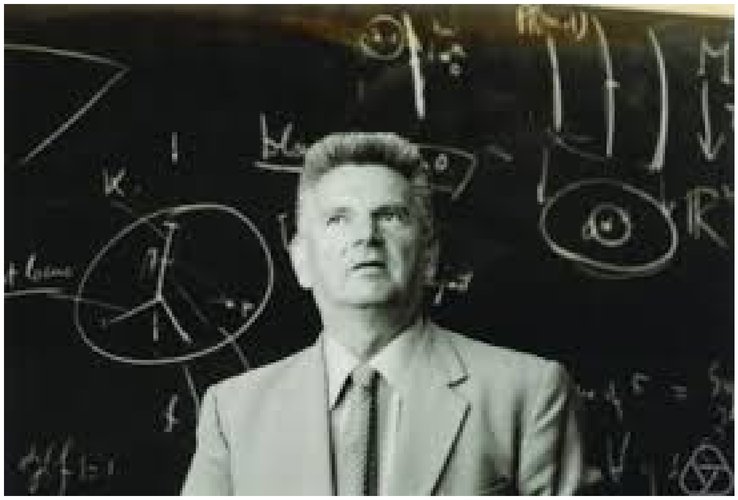


FIGURE 4. Rene Fre de Ric Thom

2.4. **The Problem.** Cobordism Theory is an area of mathematics that studies manifolds of the same dimension. It states that if the boundary of the union of two disjoint manifolds is not the same manifold then the two original manifolds are considered to be the same. Thom established a more concise approach studied these in two steps. He showed that cobordism groups were related to homotopy groups of a certain Thom space.

2.5. **Proof.** A Thom space is a topological space associated with a vector space. It enables mathematicians to lessen geometric problems to hompotopic topology problems and then hence to reduce them further to algebraic problems which can be easier solved. To construct this space:

• Let p : E → B be a vector bundle rank n, with paramount space B.

• For each b in B, ∃ a fiber Eb that is known is an n-dimensional real vector space.

• A unit sphere bundle, also of n-dimensions can be formed one way by finding the Alexandroff compactification of each fiber forming Sph(E) → B.

• The Thom space T(E) can be found from the unit sphere bundle Sph(E) by finding all the new points to a single point: . T(E) is the quotient of unit disk bundle over Sph(E) by B .

• The basepoint of T(E) is the combination of these single points, the image of Sph(E)

• If B is compact then T(E) is the Alexandroff one point compactification of E.

For example, if E is the trivial bundle, BxR^n then Sph(E)=BxSn[$^{-1}$]

We write B+ for B with a disjoint basepoint, so the Thomas space T(E) is the smash product of B+ and Sn

This work in cobordism theory then led to the idea of Thom isomorphism. Thom's work ties together differential topology and stable homotopy theory.

References

1. - Owens, A., 2021. Vaughan Jones, preeminent Vanderbilt mathematician, has died. [online] Vanderbilt University. Available at: ¡https://news.vanderbilt.edu/2020/09/09/vaughan-jones-preeminent-vanderbilt-mathematician-has-died/¿
2. - Internet Archive. 2021. A homological definition of the Jones polynomial : Stephen Bigelow : Free Download, Borrow, and Streaming : Internet Archive. [online] Available at: ¡https://archive.org/details/arxiv-math0201221¿
3. Encyclopedia Britannica. 2019. René Frédéric Thom — French mathematician. [online] Available at: ¡https://www.britannica.com/biography/Rene-Frederic-Thom¿
4. O'Connor, J., 1998. Stephen Smale - Biography. [online] Maths History. Available at: ¡https://mathshistory.st-andrews.ac.uk/Biographies/Smale/¿
5. Atiyah, M., 2004. The impact of Thom's cobordism theory. Bulletin of the American Mathematical Society, 41(03), pp.337-341.
6. En.wikipedia.org. 2021. Poincaré conjecture - Wikipedia. [online] Available at: ¡https://en.wikipedia.org/wiki/Poincar
7. Smale, S., 1961. Generalized Poincare's Conjecture in Dimensions Greater Than Four. The Annals of Mathematics, 74(2), p.391.

Knot Theory

Sean McCabe 17324923, Rebecca McKinney 17907432, Sarah Butler 17923812

7th May 2021

1 Abstract

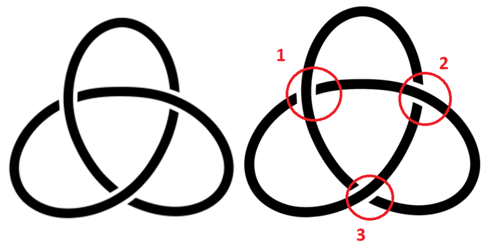
In this project we aim to explore knot theory, looking at its origins, applications and the equivalency of knots. This topic was chosen by is as it piqued our interest the most, especially that of Sean who worked with real life physical knots a lot in his job. This project has been largely influenced by the work of Carlo H. Séquin from UC Berkeley via his work with Numberphile, an educational YouTube channel.

2 What is Knot Theory?

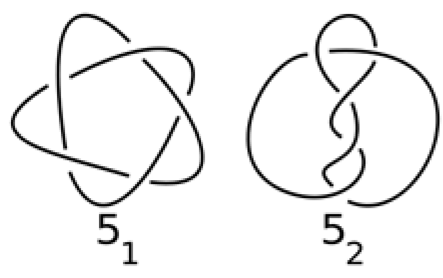
When we think of knots in the real world, we think of tying shoelaces or docking boats into the bay, but a Mathematical knot is far different than these real-world examples. To be a topological knot, there mustn't be any loose ends, and the simplest way of imagining this is a rubber band- i.e. a trivial knot. A rubber band has no knots in it, even if you attempt to knot it, the knots can be undone and it will go back to the original ring shape, with a total of zero knots (hence it is a trivial knot). In the simplest of terms, a knot is a 3-dimensional space that is homeomorphic (equivalent in topological shape) to a circle (e.g. our rubber band example). The formal definition is as follows:

> *K is a knot if there exists a homeomorphism of the unit circle C into 3-dimensional space* R^3 *whose image is K* (Crowell and Fox, 2012)

Now that we have defined what a knot is, we can then be categorize them by how many knots (or crossovers) they have; *the Crossing Number.* The Crossing Number describes the smallest number of crossings of any diagram of the knot (Crossing number (knot theory) - Wikipedia, 2021). Displayed below is the Trefoil Knot - containing 3 crossovers. Any other 3 knot will be homeomorphic to the Trefoil knot, meaning they are topologically equivalent.



As previously stated, the Crossing Number is the minimum number of crossings, so a knot has to be detangled to find the minimum amount, as sometimes they may appear to have a higher crossing number than they actually do. If a knot is detangled and found to have a crossing number of 3, then it is homeomorphic to the Trefoil knot. Similarly, there is only 1 knot with a Crossing Number of 4, but once we look into higher crossing numbers, there becomes knots with equal crossing numbers that aren't homeomorphic, e.g. the following 2 knots have a Crossing Number of 5, but are not homeomorphic to each other:



In fact, these are the only 2 knots with a Crossing Number of 5. The problem is untangling a knot in such a way to establish whether it is the knot of 5_1 or 5_2. When we begin to look at higher crossing numbers, the number of different topological knots that can be obtained becomes interesting; there are 3 knots of crossing number 6, 7 with crossing number 7, 21 crossing number 8 knots and 49 crossing number 9 knots. Here we begin to see it grow exponentially, as the number of crossing number 10 knots is 165. The great challenge is first determining whether the minimum crossing number has been found, and then establishing which knot it corresponds to- neither of which are easy tasks. Below is the table of knots up to and including crossing number 8:

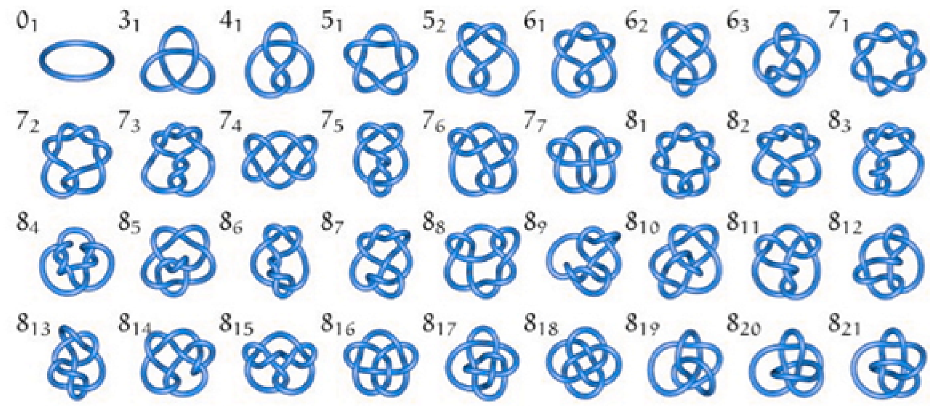
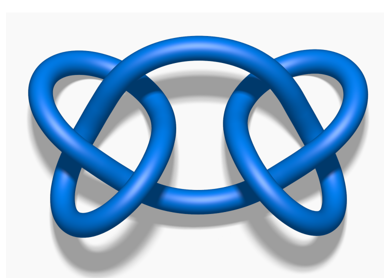


Table of knots through 8 crossings

It is also important to note that when we give a knot a minimum crossing number and identify it from the known knots, we are actually talking about Prime Knots.

3 Prime and Composite Knots

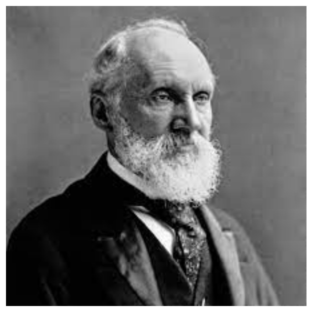
In the above examples, we have been looking at the idea of Prime Knots. A Prime knot is "indecomposable", meaning that it cannot be formed by the sum of two other non-trivial knots (Prime knot - Wikipedia, 2021). In contrast, a composite knot can be formed by combining prime knots together; in simpler terms, if you took a large knot and would be able to "cut" the knot into two smaller prime knots, then your original knot would have been composite. Below is an example of the square knot, which appears to have a crossing number of 6, but it could be split into two trefoil knots of crossing number 3.



The Square Knot

4 The Origins of Knot Theory

German mathematician Carl Friedrich Gauss was the first to make ripples in the pond of knot theory when he developed the Gauss linking integral for computing the linking number of two knots around 1800. However the person accredited with really introducing knot theory to the world was Scottish mathematician-physicist William Thomson, who was more commonly known as Lord Kelvin, in 1869. He suggested that atoms might consist of knotted vortex tubes of the ether, with different elements corresponding to different knots.



Lord Kelvin

Another Scotsman, Peter Guthrie Tait, followed this by making the first systematic attempt to classify knots. While these initial theories were eventually disproved, knot theory continued as a purely mathematical theory for around a century until a breakthrough came in 1984 from New Zealand mathematician Vaughan Jones when he introduced Jones polynomials as new knot invariants. This led Edward Witten, an American physicist to discover a connection with knot theory and quantum field theory a few years later.

Similarly, in the early 1980s, the important link between knot theory and hyperbolic geometry was made by the American mathematician William Thurston. From them humble beginnings, links to knot theory have now been made to a vast number of branches of mathematics and science (Osserman, 2016)

5 Reidemeister Moves

In the 1930's, Kurt Reidemeister showed that knots exist which are distinct from the unknot/trivial knot (Wolfram MathWorld, 2021). This was proved by showing all knot deformations can be reduced to a sequence of three different types of moves:

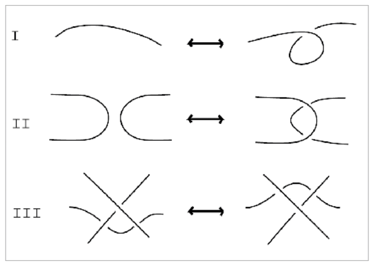
i)Twist and untwist in either direction.

ii)Move one loop completely over another.

iii)Move a string completely over or under a crossing.

These three moves that are illustrated below and are most commonly referred to as the Reidemeister moves. If two knots are equivalent it can be proved using a sequence of Reidemeister moves (Wikipedia, 2021)

This was a major development in knot theory, as it provides a framework for unknotting complicated knots in order to assess whether they are isotopic to a pre-existing knot. This is particularly helpful when seeking to establish whether or not such complicated knots are equivalent to existing prime knots which have been tabulated over the years.

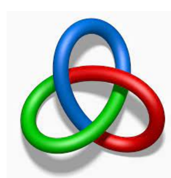


6 Knot Colourability

A strand in a knot diagram is a continuous piece that goes from one undercrossing to the next. The number of strands is the same as the number of crossings. A knot is tricolorable if each strand of the knot diagram can be colored one of three colors, subject to the following rules:

i) A minimum of two colours is used

ii) At each crossing, the three incident strands are either all the same color or all different colors



Coloured trefoil knot

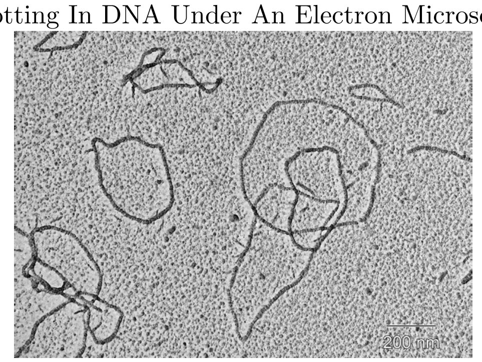
Tricolorability is an invariant under Reidemeister moves and is used to check whether knots are distinct. As the unknot is not tricolourable for example, every knot that is tricolourable is not equivalent to the unknot. (Xiaoyu Qiao, E. L.,2015)

7 Applications of Knot Theory

Knot Theory In Biology

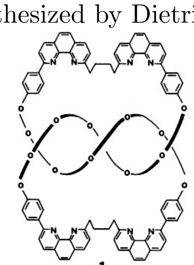
Deoxyribonucleic Acid - more commonly referred to as DNA, consists of two polynucleotide strands twisted around each other in a double helix. The tightly wound DNA is packed into chromosomes and genes which must be topologically manipulated or unpacked in order for DNA replication to occur. It is easier for replication to occur if the DNA is neatly arranged, this is accomplished with the help of enzymes called topoisomerases which can topologically manipulate the DNA. Molecular biologists are using knot theory to investigate the mechanisms that the enzymes use for DNA packing and unpacking (DNA and Knot Theory, 2021). Some experimental work with gel electrophoresis have shown there is a linear relationship between the distance of electrophoretic migration by different DNA knots and the average crossing number of the knot. This enables scientists to identify the type of knot by simply measuring its position on the gel. The exception to this rule involves right and left-handed forms of the same knot. Another process that scientists can use is taking pictures of flattened DNA with an electron microscope. Different segments of DNA are highlighted with a protein coating and the DNA can be visualised as a knot. Following this the crossing number can then be estimated(Weber, Stasiak, De Los Rios and Dietler, 2006).

Knotting In DNA Under An Electron Microscope



Over the years, the importance of knots in naturally occurring biological systems has played a significant role in motivating chemists to develop synthetic strategies for creating topologically complex molecules (Lim and Jackson, 2015). In 1989 French chemists Dietrich-Buchecker and Sauvage synthesized the first knotted compound ever made, a 124-atom molecule shaped like a trefoil- Catenane (Dietrich-Buchecker and Sauvage, 1989). Knot theory is evident in other fields such as cryptography, string theory, medicine and statistical mechanics.

Molecular trefoil knot synthesized by Dietrich-Buchecker and Sauvage.



1

8 Bibliography

Crowell, R.H. and Fox, R.H., 2012. Introduction to knot theory (Vol. 57). Springer Science and Business Media.

Dietrich-Buchecker, C. and Sauvage, J., 1989. A Synthetic Molecular Trefoil Knot. Angewandte Chemie International Edition in English, [online] 28(2), pp.189-192. Available at: ¡https://onlinelibrary.wiley.com/doi/abs/10.1002/anie.198901891¿ [Accessed 5 June 2021].

Lim, N. and Jackson, S., 2015. Molecular knots in biology and chemistry. Journal of Physics: Condensed Matter, 27(35), p.354101.

Mathworld.wolfram.com. 2021. Reidemeister Moves – from Wolfram MathWorld. [online] Available at: ¡https://mathworld.wolfram.com/ReidemeisterMoves.html¿ [Accessed 5 June 2021].

Osserman, Robert. ”Knot theory”. Encyclopedia Britannica, 17 May. 2016, https://www.britannica.com/science/knot-theory..

Tiem.utk.edu. 2021. DNA AND KNOT THEORY. [online] Available at: ¡http://www.tiem.utk.edu/ gross/bioed/webmodules/DNAknot.html¿ [Accessed 15 May 2021].

Web.cse.ohio-state.edu. 2021. [online] Available at: ¡http://web.cse.ohio-state.edu/ machiraju.1/teaching/CSE5544/Visweek2012/scivis/papers/zhang.pdf¿ [Accessed 5 June 2021].

Weber, C., Stasiak, A., De Los Rios, P. and Dietler, G., 2006. Numerical Simulation of Gel Electrophoresis of DNA Knots in Weak and Strong Electric Fields. Biophysical Journal, [online] 90(9), pp.3100-3105. Available at: ¡https://www.sciencedirect.com/science/article/pii/S0006349506724924¿ [Accessed 15 May 2021].

En.wikipedia.org. 2021. Reidemeister move - Wikipedia. [online] Available at: ¡https://en.wikipedia.org/wiki/$Reidemeister_move > [Accessed 6 June 2021]$.

En.wikipedia.org. 2021. Crossing number (knot theory) - Wikipedia. [online] Available at: $< https : //en.wikipedia.org/wiki/Crossing_number_(knot_theory) >$ [Accessed 14 May 2021]. En.wikipedia.org. 2021. Prime knot - Wikipedia. [online] Available at: ¡https://en.wikipedia.org/wiki/Prime$_knot > [Accessed 6 June 2021]$.

Xiaoyu Qiao, E. L. January 20, 2015, $< http : //web.math.ucsb.edu/ padraic/ucsb_2014_15/ccs_problem_solvi$

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

A BRIEF INTRODUCTION TO TOPOLOGICAL DATA ANALYSIS

ARIEA MCDAID, CATHERINE FRASER, AND MAEVE DUNNE

(Communicated by Julie Bergner)

Abstract. Topological data analysis (TDA) is an approach to the analysis of data-sets using techniques from topology [2]. In order to extract information from these data-sets that can be deemed high-dimensional and noisy, TDA equips us with a general framework to analyze such data in a way that is insensitive to the particular metric chosen and provides dimensionality reduction and robustness to noise. TDA is a relatively recent field that emerged from various works in applied (algebraic) topology and computational geometry during the first decade of the century. Thanks to the rise of TDA, we now have more algorithmic methods to infer, analyze and exploit the complex topological and geometric structures underlying data that are often depicted as point clouds in Euclidean spaces. The main research problem of this article is how difficult it can be to analyse Covid-19 data, with the main objective of this article being that the use of TDA can help solve this issue as it effectively represents the growth rate of Covid-19, if used correctly. The methodology behind this takes the form of researching both mapper and persistent homology and how they aid in providing a way of demonstrating such data aptly.

1. Introduction

Topological Data Analysis (TDA) is a recent and fast-growing field providing a set of new topological and geometric tools to infer relevant features for possibly complex data.[4] This field can be traced back to the pioneering works of Edelsbrunner et al. (2002) and Zomorodian and Carlsson (2005) in persistent homology and was popularized in a landmark paper in 2009 Carlsson (2009). TDA is mainly motivated by the idea that topology and geometry provide a powerful approach to infer robust qualitative, and sometimes quantitative, information about the structure of data. Although it is still rapidly evolving, TDA now provides a set of mature and efficient tools that can be used in combination or complementary to other data sciences tools.

Much of TDA is based around the notion that there is an idea of proximity between these data points. So, for example, if each data point $x = \{x_1, ..., x_n\}$ consists of n numerical values, we have an easy definition of proximity that comes from the standard Euclidean distance: this is the generalization of the standard distance in the plane

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

2020 *Mathematics Subject Classification.* Primary 55N31.

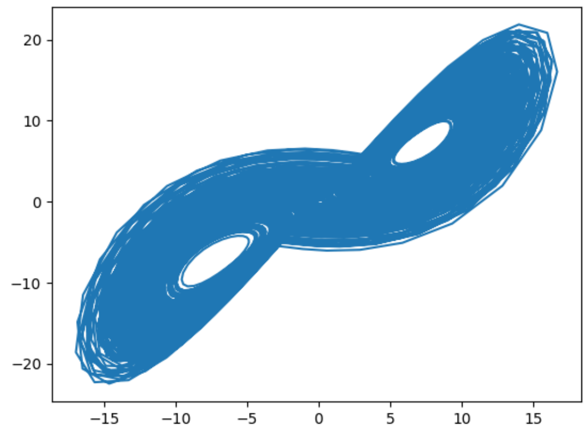


Figure 1. A 2D projection of the Lorenz system trajectory

The Euclidean distance gives a good intuitive starting place for the requirements of a generalized distance in the mathematical sense. When we have a collection of data points with a definition of a distance, we often refer to this collection as a point cloud.[1]

A "continuous" shape is built on top of the data in order to highlight the underlying topology or geometry. This is often a *simplicial complex* or a nested family of simplicial complexes, that reflects the structure of the data at different scales. Simplicial complexes can be seen as higher dimensional generalizations of neighboring graphs that are classically built on top of data in many standard data analysis or learning algorithms. Topological or geometric information is extracted from the structures built on top of the data. This can result in a full reconstruction, typically a triangulation, of the shape underlying the data from which topological/geometric features can be easily extracted, or approximations from which the extraction of relevant information requires specific methods, such as e.g. persistent homology. Figures 1, 2 and 3 demonstrate this process.

Recall, that a simplicial complex is a structure generated from such generalized connections between data-points. The simplicial complex in the figure above is generated from triangles, lines and points. Observing the graphs, we notice that a simplicial complex is able to reconstruct the topological structure of an object from discrete data. This tool can be used to define a topology for systems that inherently don't possess shape and attempt to explain them better. The simplicial complex like the one shown in the above figure, are defined only for a specific range of distances - their structure changes as you change the distance.[5]

If we are to discuss persistent homology in this article, we must first define homology itself. Homology is a classical concept in algebraic topology providing a powerful tool to formalize and handle the notion of topological features of a topological space or of a simplicial complex in an algebraic way. For any dimension k, the k-dimensional "holes" are represented by a vector space H_k whose dimension is intuitively the number of such independent features. For example,

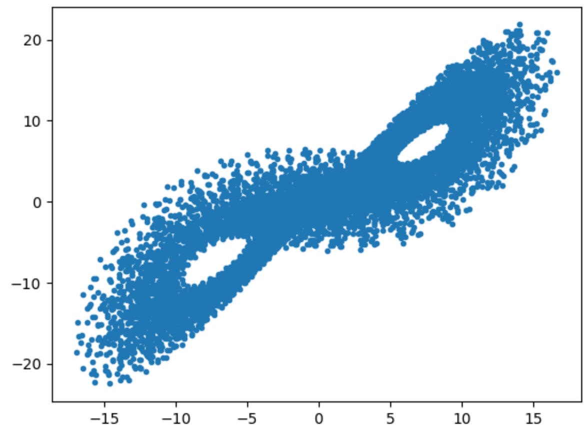


FIGURE 2. 2D Lorenz system's point cloud data

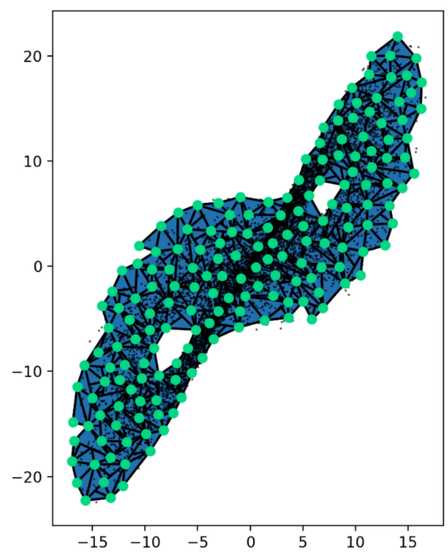


FIGURE 3. A simplicial complex generated from the point cloud data of Lorenz

the 0-dimensional homology group H_0 represents the connected components of the complex, the 1-dimensional homology group H_1 represents the 1-dimensional loops and the 2-dimensional homology group H_2 represents the 2-dimensional cavities.

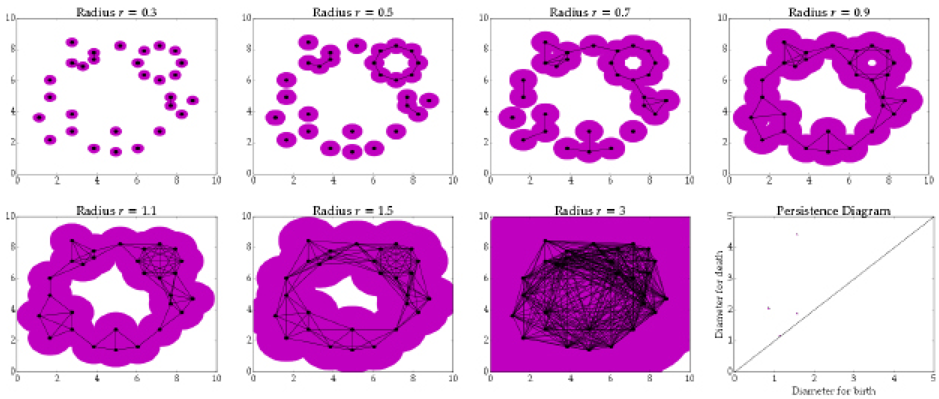


Figure 4. This is an example of using persistent homology to examine a point cloud, by constructing the Rips complex. The edge set is drawn in black in each case. The persistence diagram describes the loops in the space over the changing parameter t. Note that $t = \frac{r}{2}$. [1]

2. Persistent Homology

Persistent homology is a method of investigating the topological features of data. It is one of the most commonly used tools in TDA.[3]

As mentioned previously, simplicial complexes are a useful mechanism in investigating the shape of data. While generally graphs are used in data analysis, a lot of information can be lost here as graphs are one dimensional, i.e. they only represent the 0-dimensional data points and the 1-dimensional edges between them. Simplicial complexes allow us to represent more information, as they are k-dimensional.[1]

One way in which we can build a useful simplicial complex is through the Vietoris-Rips complex, or simply the Rips complex. Using the pre-defined definition of distance, the Rips complex for a parameter t ($t \geq 0$) is constructed such that a simplicial complex given by vertices $x_0, \ldots, x_k$ is included if every vertex is within a distance t of every other vertex in the simplex.[1] It is worth noting that this is just one way to apply persistent homology, but we will not investigate any others in this paper.

Persistent homology involves examining how the underlying structures in the data change over all possible values of t, hence the word 'persistent'. Figure 4 (above) shows an example of the Rips complex constructed for a point cloud at different parameters t. The persistence diagram given in the bottom right of the figure summarises the emergence and disappearance of loops in the space, as the parameter t changes.

The stability theorem, Cohen-Steiner, Edelsbrunner and Harer (2007), offers us an important justification for the use of persistent homology in data analysis. It states that the (Bottleneck) distance between two persistence diagrams cannot be greater than the (Hausdorff) distance between the two data sets from which the persistence diagrams arose. In context, this tells us that even if our data is effected by noise, we will still obtain an approximately correct persistence diagram.[1]

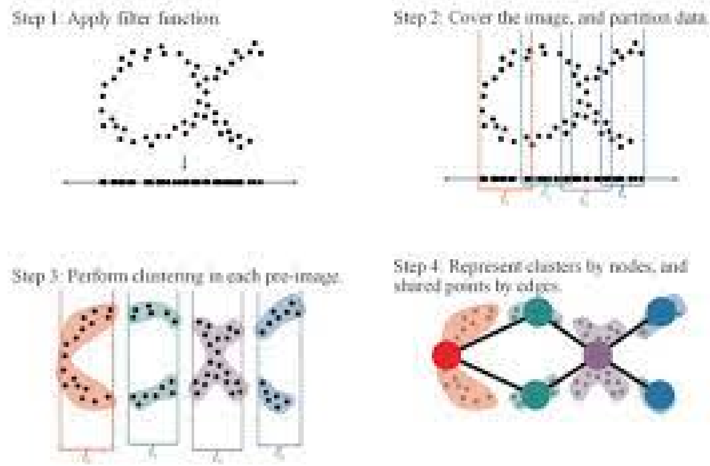


FIGURE 5. An example of the mapper algorithm being applied to a point cloud. (Text reads - Step 1: Apply filter function. Step 2: Cover the image, and partition data. Step 3: Perform clustering in each pre-image. Step 4: Represent clusters by nodes, and shared points by edges) [7]

3. The Mapper Algorithm

Mapper, stemming from Singh et al. (2007), is a topological signature that has proved very useful in applications such as biology and health sciences (Nicolau, Levine, Carlsson, 2011; Li et al., 2015; Torres et al., 2016; Nielson et al., 2015; Yao et al., 2009). It is an algorithm that uses graphs to represent the 1-dimensional structure of a data set, and as such it is a great tool for visualisation and investigation.[1] Using the nerve of covers as a way to summarize, visualize and explore data is a natural idea that was first proposed for TDA in Singh et al. (2007), giving rise to the so-called Mapper algorithm.

The mapper algorithm depends on the users choice of the so-called 'filter function', a real valued function from the data set to the space $R_n, (n \geq 1)$ i.e., a function that assigns a real number to each data point. The choice of filter function is highly dependant on the data features that one expects to call attention to [4], and the mapper algorithm's output is dependent on the user choosing a suitable filter function.

A cover of the filter function values is then chosen, and next we perform clustering. The clusters are represented as nodes on the mapper graph, and edges are included depending on the intersections between clusters.[1] A visualisation of this four step process can be seen in Figure 5 (above). In the next section, we will see an example of how the mapper algorithm can be used to analyse Covid-19 data.

4. Using The Mapper Algoritm in COVID-19 DATA

An example of TDA in practice is its use in visualizing the evolution of Covid-19 cases in England (Dlotko Rudkin, 2020). Dlotko and Rudkin used the Ball Mapper algorithm (Dlotko, 2019) to examine the relationship between the evolution of Covid-19 cases in England between March 14th and April 17th 2020, and certain socio-economic factors. The specific factors considered were population density, median age, gross domestic product (GDP), gross value added (GVA), average hours worked, and average annual gross income. The regions were determined by *NUTS* (the Nomenclature of territorial units for statistics), the hierarchal system for dividing up the geographic regions in the EU and the UK.

Using the Topological Data Analysis Ball Mapper algorithm, an abstract representation of *NUTS*3 level economic data was constructed, overlaying onto it the confirmed cases of Covid-19 in England. In doing so, this aids in understanding how the disease spreads on different socio-economical dimensions. This technique helps to visualize multiple dimensional data sets represented in the form of point clouds and therefore, helps to understand where the considered activity, in this case the infection spread, is most prevalent. The TDA Ball Mapper algorithm highlights where the characteristic space cases are particularly fast rising in terms of number of infections. Whilst regional characteristics are comparatively static, the changing nature of cases facilitates the envisioning of changing outcomes on the point cloud. A second contribution is thus to show how the relative levels of Covid-19 are changing within the space. Early work to link Covid-19 to economic conditions in China has sought data science to explain which factors are important to spread (Qiu et al., 2020), but such Machine Learning approaches are subject to overfitting criticisms. The BM (Ball Mapper) approach here is simply a representation of the data that the user interprets. There is also much to remind of other visualization methods, such as t-SNE (Maaten and Hinton, 2008) as endorsed recently in medical analysis by Linderman et al. (2019), and the original TDA mapper algorithm of Singh et al., (2007). Unlike these approaches BM does not require the construction of a full distance matrix between points as t-SNE does, and is more stable than the original mapper as BM has just one parameter and no functional choices. Learning from past studies of disease spread, the methodological advancements of TDA and BM help to explore what one may learn from the evolving English picture.[6]

5. Conclusion

In conclusion, there are many benefits of TDA, which has been successfully applied to a range of applications in the recent years.[2]

On the lines of machine learning, TDA belongs to a category of mathematical tools that aim to determine mathematical associations or patterns in data from complex systems, without claiming to understand their inner mechanisms. The difference between TDA and general Machine Learning (ML) is that TDA is specifically concerned with the analyzing of patterns or properties pertinent to the shape of the data. One advantage that TDA definitely offers is interpretability, something that ML falls short in current times.[2] TDA provides a new approach of understanding patterns in your data that are associated with its shape. Whereas ML is restricted to making possible inferences between data and shape definitions.

On the application side, many recent promising and successful results have demonstrated the interest of topological and geometric approaches in an increasing number of fields such has, e.g., material science Kramar (2013), 3D shape analysis Skraba (2010) and multivariate time series analysis Khasawneh and Munch (2016) to name a few. It is beyond the scope to give an exhaustive list of applications of TDA. On another hand, most of the successes of TDA result from its combination with other analysis or learning techniques. Therefore, it is safe to assume that one can expect to see even more applications of topological data analysis in the coming years.

References

1. Elizabeth Munch, *A User's Guide to Topological Data Analysis*, Journal of Learning Analytics, 4(2), 47-61 (2017)
`https://files.eric.ed.gov/fulltext/EJ1149275.pdf`
2. Varad Deshmuhk, *Topological Data Analysis - A Very Short Introduction*, (2019)
`https://medium.com/@varad.deshmukh/topological-data-analysis-a-very-short-introduction-611d3238a0bd`
3. Larry Wasserman, *Topological Data Analysis*, (2016)
`https://arxiv.org/pdf/1609.08227.pdf`
4. Frederic Chazal and Bertrand Michel, *An Introduction to Topological Data Analysis: Fundamental and Practical Aspects for Data Scientists*, (2021)
`https://arxiv.org/pdf/1710.04019.pdf`
5. Yiran Chen and Ismar Volic, *Topological Data Analysis Model for the Spread of the Coronavirus*, (2020)
`https://arxiv.org/pdf/2008.05989.pdf`
6. Pawal Dlotko and Simon Rudkin, *Visualising the Evolution of English Covid-19 Cases with Topological Data Analysis Ball Mapper*, (2020)
`https://arxiv.org/pdf/2004.03282.pdf`
7. Emerson G. Escolar, Yasuaki Hiraoka, Mitsuru Igami and Yasin Ozcan, *Mapping Firms' Locations in Technological Space: A Topological Analysis of Patent Statistics*, (2021)
`https://arxiv.org/pdf/1909.00257.pdf`

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

THE INSCRIBED RECTANGLE PROBLEM

(Communicated by Ryan McElhatton, Ruairi Dennehy, Ross Trearty)

ABSTRACT. We show a general structural theorem about rectangles inscribed in Jordan loops. We also show how this was found through attempting to prove Toeplitz's conjecture and discuss the latest breakthrough in the search for a definitive proof of the conjecture.

1. Introduction

This project was originally inspired by a YouTube video titled "Who Cares About Topology? The Inscribed Rectangle Problem" by 3Blue1Brown [8], watching the video is recommended for further understanding of the problem.

The inscribed rectangle problem is beautiful and deceptively simple. It is a great introductory problem for Topology students to study as it incorporates key elements of an undergraduate topology course in an intuitive and interesting manner. It asks whether for every simple closed loop, are there always four points on the loop which form a rectangle, such rectangles are called inscribed rectangles. With this project we aim to show the beautiful solution to this problem, as well as taking a look at the historical developments of the problem and the recent solution to the inscribed square problem.

2. History of the Problem

Many parts of the historical developments in this section were inspired by A Survey on the Square Peg Problem by Benjamin Matchske [6]

The inscribed rectangle problem stems from the century-old square peg problem, which has a long and interesting history. Over time, many different mathematicians became interested in the problem and worked hard to obtain a solution to different variations of the problem.

Early on, mathematicians knew that it was possible to find inscribed squares on some Jordan curves. However, they were not sure if all Jordan curves contained inscribed squares. (In topology, a Jordan curve is a non-self-intersecting continuous loop in the plane).

You may ask why mathematicians were particularly interested in quadrilaterals:

2020 *Mathematics Subject Classification.* MA342 Topology.

Inscribed pentagons, which consist of 5 points generally cannot be found on Jordan curves and Inscribed triangles, which consist of 3 points, are ubiquitous:

For any triangle $\triangle$ *and any Jordan curve* $\gamma \subset \mathbb{R}^2$, $\exists$ *an inscribed triangle on* γ *that is similar to* $\triangle$.

4 points is where things became interesting due to a recurring theme in low-dimensional topology.

In 1991 a German mathematician named Otto Toeplitz gave a talk posed the question: *"Does every continuous Jordan curve in the Euclidean Plane contain four points at the vertices of a square?"*. He claimed to have a proof for convex curves, but Toeplitz never published it if his claim was true. This question was commonly called the Square Peg Problem and it remained unsolved for over a century.

2.1. **Developments.** Early progress focused on smooth curves, where curves are infinitely differentiable functions: $f : S^1 \to \mathbb{R}^2$

In 1913, Arnold Emch solved the problem for "smooth enough" convex curves [1] by using the ideas of configuration spaces and homology. Emch published a further proof in 1915 that required a weaker smoothness condition. Emch claimed that he was unaware that Toeplitz discovered the problem in 1911. Emch published a third paper in 1916 that solved the Square problem for curves that are piece-wise analytic, meaning they are built from pieces of different functions over certain intervals that have a finite number of inflexions and other singularities.

Another important breakthrough was made in 1929 by Shnirelman, a Soviet mathematician. He solved the square peg problem for smooth Jordan Curves. His paper was published in Russian, but an extended version which corrected some errors made by Shnirelman originally, was published in 1944 [4]. Mathematicians tried to solve the problem for any continuous Jordan curve by using the same method but ended up with the following result:

Any such curve γ *is the limit of smooth Jordan curves* $\{\gamma^n\}_{n=1}^{\infty}$ *that provide increasingly good approximations to* γ. All these smooth curves γ_n contain squares. Unfortunately, these sequences of squares could shrink to point, so the proof did not hold.

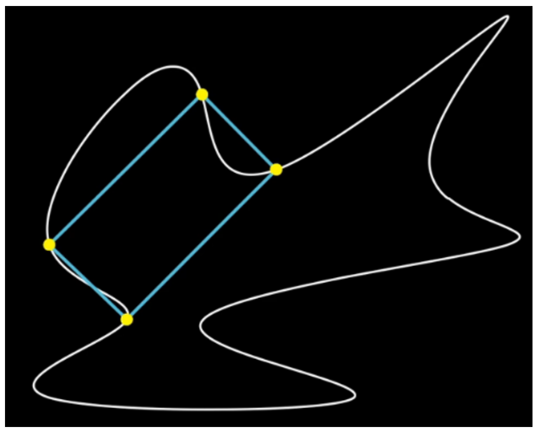
The last major breakthrough before Vaughan's inscribed rectangle proof came in 1961, when an American mathematician named Richard P Jerrard proved the square peg problem for analytic curves. His work was motivated by Kakutani's theorem that states *"every convex body has a circumscribed cube"*.

3. **The Inscribed Rectangle Problem**

Upon studying the Toeplitz conjecture: *For any Jordan curve* γ, *there exist four distinct points on* γ *such that these four points are the vertices of a square.*

H. Vaughan came closer to solving the problem than any mathematician had beforehand, and put forward the following proof for inscribed rectangles [7]:

Theorem 3.1. *Every continuous Jordan curve contains four points forming the vertices of some rectangle*



Proof. Suppose a, b, c, d $\in \gamma$ are points forming the vertices of rectangle Z

Note:

1: the line segments $a - b$ and $c - d$ are the diagonals of rectangle Z, and are of the same length ($||a - b|| = ||c - d||$)

2: the midpoint can be found by intersecting either line segment ($(a + b)/2 = (c + d)/2$

We now define a function $F : \gamma \text{ x } \gamma \to \mathbb{R}^3$ as $F(a, b) = ((a + b)/2, \; ||a - b||)$

Taking the sets of vertices that make up our line segments as ordered pairs (i.e $\{a, b\} \neq \{b, a\}$

We can see that $\gamma \text{ x } \gamma \approx S^1 \text{ x } S^1$ is in fact a torus. This means that each individual point on this torus corresponds to a unique combination of points on the Jordan curve.

If a, b, c, d $\in \gamma$ form a rectangle, then it follows that $F(a, b) = F(c, d)$. Meaning that F is never injective.

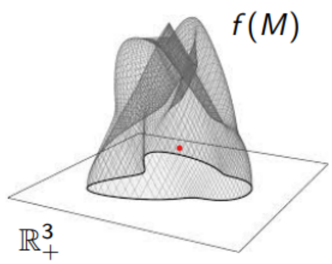
To fix this problem we need to look at the unordered pairs of points in γ , which compose the set M (where $\{a, b\} = \{b, a\}$)

The unique combinations of points on the Jordan curve can now be represented on a Mobius Strip! Where the boundary highlighted in yellow represents unordered pairs in the form $\{a, b\}$ where $a = b$. Mathematically, this means that our function F induces a map f of the mobius strip onto a 3-dimensional plane $\mathbb{R}^3$ (i.e $f : M \to \mathbb{R}^3_+$)

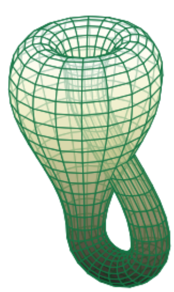
The boundary of M, shaded in yellow on the diagram and denoted δM is sent directly to γ in $\mathbb{R}^2$ x $\{0\}$ because δM represents all of the unordered pairs of the form $\{a, a\}$ (Note that $a = b$) where $f(\{a, a\}) = ((a + a)/2, ||a - a||) = (a, 0)$

We claim that $f : M \to \mathbb{R}^3$ is not injective. Non-injectivity would imply that $\exists$ points a, b, c, d $\in \gamma$ s.t $\{a, b\} \neq \{c, d\}$ and $f(\{a, b\}) = f(\{c, d\})$. These points form the vertices of a rectangle (Note that a, b, c, d $\notin \delta$M.)

Finally we need to cover the claim that $f : M \to \mathbb{R}^2$ x $\mathbb{R}_{\geq 0}$ is not injective. For contradiction, we will assume that f is injective. This would then imply that $f(M)$ is an embedded mobius strip in $\mathbb{R}^3_+ = \mathbb{R}^2$ x $\mathbb{R}_{\geq 0}$. With a boundary represented by $f(\delta$ M)=γ in $\mathbb{R}^2$ x 0.

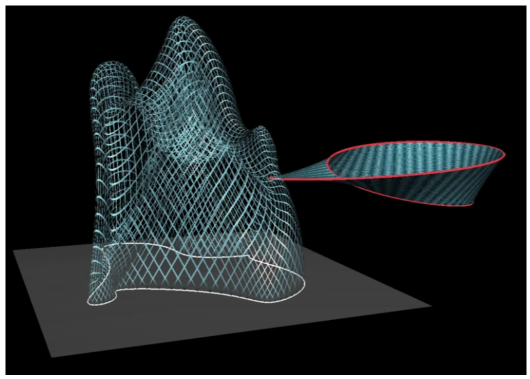


If we then take the mirror, $-f(M) \subset \mathbb{R}^3_-$ and glue this along $f(M) \subset \mathbb{R}^3_+$ we form a klein bottle, embedded in $\mathbb{R}^3$, meaning that there are no self intersections. However this is a contradiction, as there is a well known theorem in topology that *a klein bottle cannot be embedded in* R^3



□

3.1. **Summary of Proof.** We have shown that there is a one-to-one mapping between points on the mobius strip and these correspond to unique, unordered pairs of points on the Jordan Curve. Specifically at the boundary δM, these points lie directly on the loop when mapped in $\mathbb{R}^3$. As such, when we map the mobius strip onto the plane it is intuitively clear (see image below [8])



that there must exist at least one intersection to allow us to map the boundary (δM) onto the loop in the plane ($\mathbb{R}^3$). Where this intersection occurs represents an inscribed rectangle on the Jordan curve, and because it will always occur, we can say that for any continuous Jordan curve we can find at least one inscribed rectangle.

4. **The Square Peg Problem**

In 2020 two mathematicians, Joshua Evan Greene and Andrew Lobb came one step closer to a proof of Toeplitz conjecture. They showed that for all rectangles on $\mathbb{R}^2$ there exists a rectangle of the same ratio with it's vertices on any smooth Jordan curve. The following is a discussion of their the proof [9]:

Theorem 4.1. *For every smooth Jordan curve γ and rectangle R in the Euclidean plane, there exists a rectangle similar to R whose vertices lie on γ.*

Greene and Lobb's proof built on Vaughan's work but it also combined several additional results, In 2019 Cole Hugelmayer showed that inscribed rectangles in a smooth Jordan curve attains at least one third of all aspect ratios [10]. He done this by taking the Möbius strip parameterised in Vaughnan's proof and embedding it in four-dimensional space.

Take two points p, q on the Jordan curve. Each point on the Möbius strip in four-dimensional space is of the form (a,b,c,d) such that
a,b are the xy co-ordinates of the midpoint of the line segment joining p and q in $\mathbb{R}^2$.
c is the distance between the two points ($c = ||p - q||$).
d is the angle between the line pq and the x-axis in.

Hugelmeyer explained how to rotate the Möbius strip in four-dimensional space so that a,b,c were constant. The rotation only varied d. The Möbius strip can be rotated in this way through 2π radians, and he proved that one-third of those rotations yield an intersection between the original and the rotated copy. This fact turns out to be equivalent to saying that on a smooth Jordan curve, you can find rectangles with one-third of all possible aspect ratios.

Greene and Lobb expanded Hugelmeyer's discovery by focusing on a certain type of four-dimensional space in which to embed the Möbius strip. Their proof accounts for all aspect ratios, it involves embedding the strip in a four-dimensional symplectic space. This ensured that all rotations in this space yielded an intersection with the original Mobius strip. To show this they used the fact that although it is possible to embed a Klein bottle in four-dimensional space that does not intersect itself, it's impossible to do so in four-dimensional symplectic space.

Two Möbius strips that intersect each other are equivalent to a Klein bottle, which intersects itself in this type of space. If a rotation of the Möbius doesn't intersect the original copy, we have a Klein bottle that doesn't intersect itself. But such a Klein bottle is impossible in four-dimensional symplectic space. Hence we have a contradiction. Therefore, every possible rotation of the embedded Möbius strip must also intersect itself. This means every smooth Jordan curve must contain sets of four points that can be joined together to form rectangles of all aspect ratios including 1:1 which satisfies the square peg problem for smooth Jordan curves.

5. **Bibliography**

[1] A. Emch, Some Properties of Closed Convex Curves in a Plane , American Journal of Mathematics, Vol. 35, No. 4 (Oct., 1913), 407–412.

[2] A. Emch, On the Medians of a Closed Convex Polygon, American Journal of Mathematics, Vol.37 (1915), 19–28

[3] A. Emch, On Some Properties of the Medians of Closed Continuous Curves Formed by Analytic Arcs , American Journal of Mathematics, Vol. 38, No. 1 (Jan., 1916), 6–18

[4] Lev G. Schnirelman, On some geometric properties of closed curves (in Russian), Usp. Mat. Nauk, 10:34–44, 1944.

[5] Richard P. Jerrard. Inscribed squares in plane curves. Trans. Amer. Math. Soc., 1961, 98: 234–241.

[6] B. Matschke, A survey on the Square Peg Problem, Notices of the AMS, Vol. 61, No. 4, 2012.

[7] H. Vaughan, Rectangles and simple closed curves, Lecture, Univ. of Illinois at Urbana-Champagne.

[8] 3Blue1Brown, 2017, Who Cares About Topology? The Inscribed Rectangle Problem. Available from: https://www.youtube.com/watch?v=AmgkSdhK4K8t=869s

[9] Joshua Evan Greene, Andrew Lobb, The Rectangular Peg Problem, 2020, arXiv:2005.09193

[10] Cole Hugelmayer, Inscribed rectangles in a smooth Jordan curve attain at least one third of all aspect ratios, 2019, arXiv:1911.07336

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

FIELDS MEDAL

EOIN MULVIHILL, KAREN GALLEN, AND KATE CAMPBELL

(Communicated by Julia Begner)

Abstract. The Fields Medal is a prestigious mathematical award awarded to mathematicians worldwide. It was first awarded 85 years ago and has been most recently awarded in 2018. The medal is awarded to between two and four mathematicians under the age of 40 at the International Mathematical Union (IMU) congress every four years. It has been won by 60 individuals over a broad horizon of topics, many based on topology. John Milnor, Stephen Smale, William Thruston, Grigori Perelman and Maryam Mirzakani, the first and currently only female recipient, are medallists receiving the award for their research on topology. There are many landmarks over the 85 years the medal has been awarded, including the youngest winner aged 27 and protests and refusals surrounding the award.

1. Introduction

The Fields Medal is awarded every four years at the International Congress of the International Mathematical Union (IMU) to between two and four mathematicians, under 40 years of age, for outstanding or seminal research.

The Fields Medal is regarded as one of the highest honours a mathematician can receive and is often referred to as the mathematical equivalent of the Nobel Prize. However, there are several key differences such as the Fields medal is granted only every four years, the number of awards is limited to four and the medal is only granted to mathematicians under the age of 40, rather than to more senior scholars.[1]

The Fields Medal Committee is chosen by the Executive Committee of the International Mathematical Union and is normally chaired by the IMU President. The name of the Chair of the Committee is made public, but the names of other members of the Committee remain anonymous until the award of the prize at the Congress. It is asked to choose at least two, with a strong preference for four, Fields Medallists, and to have regard in its choice to representing a diversity of mathematical fields.

The name of the reward is in honour of the Canadian Mathematician, John Charles Fields. The medal originated from surplus funds raised by Fields, who was a professor of mathematics and the University of Toronto at that time. As

Received by the editors 03/05/21.
2010 *Mathematics Subject Classification.* Primary 01-XX Secondary 01Axx.

well as funding the award, he designed the medal himself. The monetary reward currently stands at CA$15,000 and is now funded by a trust originally set up by James Charles Fields at the University of Toronto.[2]

The Fields Medal was first awarded in 1936 to Finnish mathematician Lars Ahlfors and American mathematician Jesse Douglas and since 1950, it has been awarded every four years. In all, sixty people have been awarded the Field's Medal to give recognition and support to younger mathematical researchers who have made major contributions. A significant amount of the award winners have received the medal for their work in the topological field, such as John Milnor, Stephen Smale, William Thurston, Grigori Perelman and Maryam Mirzakhani which will be discussed in this article. The award has had many milestones in its existence since 1936, including the youngest winner aged 27 and protests and refusals surrounding the award.

2. John Milnor

John Milnor, famous from his work as a mathematician and topologist, received the Fields Medal in 1962 for his study of differential topology.

Milnor was born in Orange, New Jersey, U.S. on February 20, 1931 and later attended Princeton University to complete his A.B. in mathematics in 1951 and furthered his studies by receiving his Ph.D. in mathematics in 1954. Milnor's undergraduate thesis was titled "Link groups' and his doctoral dissertation focused on "Isotopy of links". During his undergraduate years Milnor was named a Putnam Fellow of Princeton University in 1949 and 1950. Milnor started his own career as a professor of Princeton after successful completion of his doctorate. Milnor was a professor at the Institute for Advanced Study from 1970 to 1990, after several years varying between alternative institutes. In 1989, Milnor became the director of the Institute for Mathematical Sciences at the State University of New York, Stony Brook. Milnor's wife, Dusa Mc Duff, has also had a successful career as a mathematician, with an academic focus on topology, and became the first recipient of the Ruth Lyttle Satter Prize in Mathematics. She is a professor of mathematics in Barnard College, New York[3].

In Stockholm 1962 Milnor received the Fields Medal at the International Congress of Mathematicians. Milnor was awarded the medal for his discovery of differential topology. At the beginning of the 20th century a geometric approach to topology was popular, but an algebraic approach in the 30's and 40's overcame the geometric focus. Milnor reignited the interest in geometric topology by his discovery of the 28 differential structures for S7, the seven-dimensional sphere, an active discovery in the creation of differential topology. Milnor discovered that there was more than one smooth structure on S7, which he termed "exotics spheres". An exotic sphere is a n-dimensional differential manifold which is homoeomorphic but not diffeomorphic to the standard n-sphere Sn. Milnor showed that there are at least seven differentiable structures on the 7-sphere. In 1963 Milnor collaborated with a French mathematician, Michel Kervaire, and computed the number of exotic spheres for dimensions greater than four[4].

John Milnor's discovery of the exotic spheres led to his awarding of the Fields Medal in 1962. He has had a successful career in respect to both teaching and research, being one of the only five to have been awarded the Fields Medal, the Abel Prize and the Wolf Prize.

3. Stephen Smale

Stephen Smale, born July 15, 1930, Flint, Michigan, U.S. is an American mathematician who was awarded the Fields Medal in 1966 for his work on topology in higher dimensions.

From 1948 to 1956 he attended the University of Michigan, obtaining B.S., M.S., and Ph.D. degrees in mathematics. As an instructor at the University of Chicago from 1956 to 1958, Smale achieved notoriety by proving that there exists an eversion of the sphere. He then cemented his reputation with a proof of the Poincaré conjecture for all dimensions greater than or equal to five, published in 1961; Smale was awarded a Fields Medal in 1966 for his work on the generalized Poincaré conjecture. This conjecture, one of the famous unsolved problems of twentieth-century mathematics, asserts that a simply connected closed 3-dimensional manifold is a 3-dimensional sphere. Smale showed that any closed n-dimensional manifold which is homotopy equivalent to the n-sphere must be the n-sphere when n is at least 5.[5]

Smale's work was remarkable in that he bypassed dimensions three and four to resolve the problem for all higher dimensions. In 1961 he followed up with the h-cobordism theorem, which became the fundamental tool for classifying different manifolds in higher-dimensional topology.

In 1998 he compiled a list of 18 problems in mathematics to be solved in the 21st century, known as Smale's problems. This list was compiled in the spirit of Hilbert's famous list of problems produced in 1900. With some of these problems including Poincaré conjecture being designated to the Millennium Prize Problems by the Clay Mathematics Institute.[6]

Smale has received many honours for his work. In addition to the Fields Medal. He was awarded the Veblen Prize for Geometry by the American Mathematical Society in 1966 for his contributions to various aspects of differential topology.

4. William Thurston

William Thurston, famous from his work as a mathematician and topologist, received the Fields Medal in 1982 for his study of topology in 2 and 3 dimensions.

William Thurston was born in Washington D.C. in 1946. He received his bachelor's degree from New College (now known as the New College of Florida) in 1967. For his undergraduate thesis he developed an intuitionist foundation for topology.

He received his doctorate in mathematics from the University of California, Berkeley in 1972. His dissertation was on Foliation of Three-Manifolds which are Circle Bundles. Thurston then spent a year at the Institute for Advanced Study, another year at MIT as an Assistant Professor. In 1974, he was appointed Professor of Mathematics at Princeton University where he remained until 1991.[7]

In 1982 it was announced that Thurston was to receive the Fields Medal and he was awarded it at the International Congress of Mathematicians in 1983 for his "Revolutionized study of topology in 2 and 3 dimensions, showing interplay between analysis, topology, and geometry. Contributed idea that a very large class of closed 3-manifolds carry a hyperbolic structure." He extended geometric ideas from the theory of two-dimensional manifolds to the study of three-dimensional manifolds. His geometrization conjecture says that every three-dimensional manifold is locally isometric to just one of a family of eight distinct types. Thurston also took up ideas about the discrete isometry groups of hyperbolic three-space, first investigated by Henri Poincaré and later studied by Lars Ahlfors. Deformations of these groups were studied by Thurston, and further advances in quasi-conformal maps resulted.[8]

Thurston's study led to many more discoveries. With Thurston's work, in 2006 Grigori Perelman won the Fields medal for the first proof published of the Poincaré conjecture in three dimensions, which was a major unresolved part of Thurston's geometric conjecture and had challenged mathematicians for 100 years. In addition, cosmologists have drawn on Dr. Thurston's discoveries in their search for the shape of the universe. Thurston's work even inspired the designer Issey Miyake's 2010 ready-to-wear collection, a colourful series of draped and asymmetrical forms.

Thurston went on to become director of the Mathematical Sciences Research Institute at Berkeley in 1992. He later taught at the University of California, Davis from 1996 to 2003 and at Cornell University from 2003 until his death in 2012. Thurston's other awards include the 2005 American Mathematical Society (AMS) Book Prize for his book "Three-dimensional Geometry and Topology." In 2012 he won the AMS Leroy P. Steele prize for a Seminal Contribution to Research, one of the highest distinctions in Mathematics.

5. Grigori Perelman

Grigori Perelman is a Russian mathematician who was born on 13 June 1966 in Leningrad, the Soviet Union to Russian-Jewish parents. He made great contributions to geometric topology and Riemannian geometry and was awarded the Fields Medal in 2006 for his work on the Poincaré conjecture and Fields medalist William Thurston's geometrization conjecture. But he declined the award, stating: "I'm not interested in money or fame; I don't want to be on display like an animal in a zoo."[9]

From 1995 to November 2002, Perelman worked alone on the Poincaré's Conjecture, cutting off nearly all contact with the mathematics community. In these seven years, Perelman was able to overcome the difficulties that crushed Hamilton's hopes of finding the proof. First, he showed the assumption that the curvature is uniformly bounded was correct, because in the particular space of the proof, it simply is always the case. Second, he showed that the singularities would always appear in the same precise case (when the flow would grow too rapidly), and conceived a function that would be effective against all of them. He even proved that some of the singularities Hamilton had identified would simply never occur.

Poincaré conjecture
The Poincaré conjecture, proposed by French mathematician Henri Poincaré in 1904, was one of the key problems in topology. Any loop on a 3-sphere—as exemplified by the set of points at a distance of 1 from the origin in four-dimensional Euclidean space—can be contracted into a point. The Poincaré conjecture asserts that any closed three-dimensional manifold, such that any loop can be contracted into a point, is topologically a 3-sphere. The analogous result has been known to be true in dimensions greater than or equal to five since 1960. But the case of three-manifolds turned out to be the hardest of them all. By topologically manipulating a three-manifold, there are too few dimensions to move ”problematic regions” out of the way without interfering with something else.[10]

The most fundamental contribution to the three-dimensional case had been produced by Richard S. Hamilton. Hamilton reached a dead end when he could not show that the manifold would not snap into pieces under the Ricci flow. Perelman's decisive contribution was to show that the Ricci flow did what was intended and that the impasse reflected the way a three-dimensional manifold is made up of pieces with different geometries.

After nearly a century of effort by mathematicians, Grigori Perelman presented a proof of the conjecture in three papers made available in 2002 and 2003 on arXiv The Poincaré conjecture, before being proved, was one of the most important open questions in topology. In 2000, it was named one of the seven Millennium Prize Problems, for which the Clay Mathematics Institute offered a $ 1 million prize for the first correct solution. Perelman's work survived review and was confirmed in 2006, leading to his being offered a Fields Medal, which he declined. Perelman was awarded the Millennium Prize on March 18, 2010 but turned down the prize, saying that he believed his contribution in proving the Poincaré conjecture was no greater than Hamilton's. As of 2021, the Poincaré conjecture is the only solved Millennium problem. [11]

6. Female recipients

To date there has been sixty recipients of the Fields Medal since it was first awarded in 1936. It is a reputable award recognised worldwide, but has only been awarded to a female once. In 2014 Maryam Mirzakhani received the award, Mirzakhani was the first and only woman to date that has received the award.

Mirzakhani, born in 1977 in Iran, made an impact from a young age becoming the first Iranian female to win two gold medals in the International Mathematical Olympiad, achieving a perfect score[12]. Mirzakhani was one of two survivors of a national tragedy in Iran when a bus transporting a team of gifted individuals and Olympiad competitors fell off a cliff in 1998. In 1999, she obtained her Bachelor of Science in mathematics in Sharif University of Technology, receiving recognition from the American Mathematical Society for her discovery of a simple proof for a theorem of Schur during her time in Sharif. Mirzakhani received a PhD from Harvard in 2004, under the guidance of 1998 Fields Medallist Curtis T. McMullen.

In 2014, Maryam Mirzakhani was awarded the Fields Medal for "her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces" in Seoul, South Korea. Her work on Riemann surfaces and their moduli spaces covers a broad variety of disciplines; topology, hyperbolic geometry, complex analysis and dynamics. A Riemann surface is a one-dimensional complex manifold. Such surfaces were discovered to be classified topologically. Mirzakhani worked on closed geodesics on a hyperbolic surface (a closed curve that cannot be shortened by deformation), moduli space and geometry of a moduli space, which all led to her Fields Medal award.

Mirzakani was to remain a key leader to discover further revelations in the field of moduli space but sadly passed away in 2017, aged 40, after a battle with breast cancer. The International Council of Science declared 12 May, Mirzakhani's birthday, as International Women in Mathematics Day[13].

7. Landmarks

Since it was first awarded, the Fields Medal has had many landmarks and disruptions in its course. The medal was first awarded in 1936 to a Finnish recipient Lars Ahlfors and American Jesse Douglas. It was not awarded for a further fourteen years in 1950, and has been given out every four years since then.

The youngest recipient of the award was French mathematician, Jean-Pierre Serre, aged 27 who won the award in 1954 for his major results on the homotopy of groups of spheres.
The Soviet government took account for many disturbances for the acceptance of awards, in 1966 Alexander Grothendieck boycotted the award ceremony to protest Soviet military actions that were occurring in Easter Europe. Sergie Novikov and Grigory Margulis were unable to attend the congress in Nice in 1970 and Helsinki in 1978, respectively, due to restrictions placed upon them by the Soviet government.

The 1982 congress was due to be held in Warsaw but it had to be postponed to 1983 due to a martial law being introduced in Poland in 1981. The government of Poland restricted everyday life of individuals during a political disruption, forcing the IMU congress to postpone.
1990 saw the first physicist win the award, Edward Witten surprised the mathematical community by applying a physical insight which led to new and deep mathematical theorems. Shortly after, in 1998, Andrew Wiles received the first silver plaque for his proof of Fermat's Last Theorem, thought to be given due to his age exceeding the age limit of forty for the award.

In the 21st century, Grigori Perelman refused his Fields Medal and did not attend the congress for his proof of Poincaré conjecture in 2006. In 2014, Maryam Mirzakhani became the first woman and first Iranian to win the Fields Medal, alongside Artur Avila becoming the first South American and Manjul Bhargava the first Indian recipient[1].

8. Conclusion

In the absence of a Nobel Peace Prize in mathematics, the Fields Medal recognises and rewards the outstanding research and breakthroughs of young mathematicians. It encourages innovative thinking and the move into uncharted territories. As well as that, the award has a knock on effect of success, with many winners of the award basing their research on previous Field Medal winners.

References

1. Wikipedia: The Fields Medal
   en.wikipedia.org/wiki/Fields_Medal

2. Britannica: Fields Medal mathematics award
   https://www.britannica.com/biography/Enrico-Bombieri

3. Wikipedia: John Milnor
   https://en.wikipedia.org/wiki/John_Milnor

4. Britannica: John Willard Milnor
   https://www.britannica.com/biography/John-Willard-Milnor

5. Wikipedia: Stephen Smale
   https://en.wikipedia.org/wiki/Stephen_Smale

6. Britannica: Stephen Smale American mathematician
   https://www.britannica.com/biography/Stephen-Smale

7. Wikipedia: William Thurston
   https://en.wikipedia.org/wiki/William_Thurston

8. Britannica: William Paul Thurston American mathmatician
   https://www.britannica.com/biography/William-Paul-Thurston

9. Wikipedia: Grigori Perelman
   https://en.wikipedia.org/wiki/Grigori_Perelman

10. Wikipedia: Poincaré conjecture
    https://en.wikipedia.org/wiki/Poincaré_conjecture

11. Britannica: Grigori Perelman American mathmatician
    https://www.britannica.com/biography/Grigori-Perelman

12. New Scientist: Iranian woman wins math's top prize, The Fields Medal
    https://www.newscientist.com/article/dn26044-iranian-woman-wins-maths-top-prize-the-fields-medal

13. Wikipedia: Maryam Mirzakhani
https://en.wikipedia.org/wiki/Maryam_Mirzakhani

Co. Westmeath, Ireland

Co. Donegal, Ireland

Co. Wicklow, Ireland

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

WINNERS OF THE FIELDS MEDAL

GORDON O'CONNOR AND CORMAC DEIGNAN

(Communicated by David Futer, Temple University)

Abstract. In this article, we will briefly review what the Fields Medal is, and consider six topologists who have been awarded the Fields Medal.

1. Introduction

There are certain aspirations and goals that many great mathematicians strive to achieve. One of these goals that very few young geniuses accomplish is the receiving of the Fields Medal award. It is an award that drives innovation and advancements in mathematics and in this project we will look at which topologists reached this very ambitious goal and what they did to deserve it.

2. Fields Medal

The Fields Medal is considered by most to be one of the most prestigious awards that a mathematician can be honoured by. Only 60 people have ever received this award[1] since it was first conceived 85 years ago in honour of John Charles Fields who was a main founder of the award and who helped support the monetary component of the award. Every four years, up to four candidates are chosen for this prize at the International Mathematical Union(IMU) and thus far six topologists have brought this medal to varying home countries including Russia, USA, England and France.

Figure 1. Fields Medal [7]

Received by the editors 07-June-2021.
2010 *Mathematics Subject Classification.* 57M25.

3. Rene Thom

Rene was born in France in the year 1923. He was originally educated at the Lycee Saint-Louis and the Ecole Normale Supérieure in Paris from which he continued on to receive his PhD in 1952 from the University of Paris. From here on , Rene had a very successful career in which he won the very first Fields Medal awarded to a topologist in 1958[2],went on to teach at the University of Grenoble , won the Brouwer Medal in 1970 and also won the Grand Prix Scientifique de la Ville de Paris in 1976.

We are mainly concerned however, with what Rene did to outshine his competition for the Fields Medal in 1958. Rene won the Fields Medal that year based on his written work on the foundations of cobordism theory. "Cobordism" literally means "jointly bound". Cobordism is a much coarser equivalence reaction than homeomorphism of manifolds (a manifold is a topological space that locally resembles Euclidean space near each point) and is easier to study(supposedly!). As shown in the figure on the right, M and N are cobordant if they jointly bound a manifold. Cobordisms are central objects of study in geometric topology and in algebraic topology. In geometric topology they are closely connected with Morse theory and in the study of high dimensional folds like surgery theory which can be described as removing an inner sphere from an outer manifold. These original concepts written by Rene Thom lead to many different growths down the line in mathematics.

4. Michael Atiyah

Michael Atiyah was the very next topologist to win the Fields Medal after Reme[3, 4]. He was born in 1929 and passed away as recently as January of 2019. He spent most of his academic life between Oxford and Cambridge whilst also going to the United States at the Institute of Advanced Study however he was born in and grew up in Sudan and Egypt. What made Michael stand out in the race for the Fields Medal in 1966 was his work on K-theory.He was one of the pioneers on K-theory at the time alongside Alexandre Grothendieck and Friedrich Hirzebruch.

Michael and his fellow mathematicians first saw K-theory as a way to work with higher dimensional twistings of something that resembles a Mobius band.A Mobius band can be shown by twisting a strip of paper one time as shown on the leftThis shows a rank one vector bundle over a circle. They later used their K-theory to improve lower bounds in comparison to ordinary cohomology for the James number and also used it as a more efficient way to give a solution to the Hopf invariant one problem. It has all led to a very wide range of uses in many different fields of mathematics such as number theory, algebraic geometry and even including operator theory. It can now be seen as the study of different types of invariants of large matrices.

5. Stephen Smale

This Fields Medal winner won [5, 6] in the very same year as Michae Atiyah.He was born only a year after Atiyah however and led a relatively boring life in comparison. He was born in America, grew up there and also stayed on the Mathematics faculty in the University of California for more than three decades. He first showed his genius when he proved that there does exist an eversion of a sphere (it is possible to turn a sphere inside out) and cemented it when he followed up in 1961 with his theorem on h-cobordism.It later became the fundamental tool in classifying manifolds in higher-dimensional topology.

6. Sergei Novikov

Sergei Novikov is a Russian mathematician, originally from Nizhny Novgorod, and was born there on the 20th of March 1938. Novikov earned himself a PhD and Doctorate of Mathematics from the V.A. Steklov Institute of Mathematics in Moscow, receiving his doctorate in 1965. In 1970 he was awarded the Fields Medal for his contributions to topology [8].

Novikov was one of the pioneers of surgery theory, a collection of techniques that are used to solve geometric topological problems. Surgery theory is used to classify high-dimensional manifolds, i.e. topological spaces that resemble real n-dimensional Euclidean spaces. His work on proving the topological invariance of the rational Pontryagin classes and the Novikov conjecture is what he was recognised for when he was awarded the Fields Medal.

The Novikov Conjecture is one of the most important unsolved problems in the field of topology [9]. The conjecture deals with homotopy invariance of certain polynomials in the Pontryagin classes, arising from the fundamental group. Pontryagin classes are certain characteristic classes of real vector bundles. Note that a characteristic class is a method of associating each principle bundle of X with a cohomology class of X, where a principle bundle is an object that formalises some of the essential features of the Cartesian product of a space X with a group G, and a cohomology is a general term for a sequence of abelian groups associated with topological space. The Pontryagin classes all lie in cohomology groups with degrees of a multiple of 4.

According to the Novikov conjecture, the higher signatures (certain numerical invariants of smooth manifolds) are homotopy invariants. The conjecture is true for all finitely generated abelian groups, but whether or not it holds true for all groups has yet to be proven.

7. William Thurston

William Paul Thurston was an American mathematician originally from Washington D.C. and was born there on the 30th of October 1946. Thurston earned a doctorate in Mathematics from the University of California Berkely in 1972,wher his dissertation was on Foliaths of Three-Manifolds which are Circle bundles. He workedas a professor of mathematics and computer science at Cornell University from 2003 until his death in 2012. In 1982, he was awarded the Fields Medal for his contributions to the study of 3-manifolds [10].

The bulk of Thurston's early work was spent developing foliation theory. Some of his more notable contributions to the field include the proof that every Haelfliger structure on a manifold can be integrated to a foliation, and the proof that the cohomology of the group of homeomorphisms of a manifold is the same whether the group is considered with its discrete topology or its compact-open topology (a topology defined on the set of continuous maps between two topological spaces). However, it was his later work (post 1975) that earned him a Fields Medal in 1982.

8. John Milnor

John Willard Milnor is an American mathematician originally from Orange, New Jersey and was born there on the 20th of February 1931. Thurston earned a Ph.D. in Mathematics from the Princeton University in 1954, where he completed his doctoral dissertation on 'Isotopy of Links'. He worked as a professor at the Institute of Advanced Study in Princeton from 1970 to 1990. In 1962, he was awarded the Fields Medal for proving that 7-dimensional spheres can have several different structures, which ultimately lead to the creation of the field of differential topology [11].

Milnor first published the proof in 1956. A 7-sphere is a topological space that is homeomorphic to a standard 7-sphere, which is the set of points in 8-dimensional euclidean space that are situated at a constant distance r from a fixed point called the centre.

9. Conclusion

It is clear that topology is an important field of mathematics when a prize as prestigious as the Fields Medal is frequently awarded to the great minds working in this field. From Rene Thom's award in 1958 all the way up to present day, the history of the Fields Medal shows us the fascination some of the worlds best mathematicians have had with topological problems and questions. One can only imagine what future discoveries and innovations in the field of topology will merit this award in the years to come.

References

1. List of Fields Medal Winners
`https://stats.areppim.com/listes/list`$_{f}ieldsxmedal.htm$
2. Encyclopedia Britannica: Rene Thom
`https://www.britannica.com/biography/Rene-Frederic-Thom`
3. Encyclopedia Britannica: Michael Atiyah
`https://www.britannica.com/biography/Michael-Francis-Atiyah`
4. Wikipedia: Michael Atiyah
`https://en.wikipedia.org/wiki/Michael`$_{A}tiyah$
5. Encyclopedia Britannica: Stephen Smale
`https://www.britannica.com/biography/Stephen-Smale`
6. Wikipedia: Stephen Smale
`https://en.wikipedia.org/wiki/Stephen`$_{S}male$
7. Wikipedia: Fields Medal Image
`https://en.wikipedia.org/wiki/Fields`$_{M}edal$
8. Wikipedia: Sergei Novikov
`https://en.wikipedia.org/wiki/Sergei`$_{N}ovikov_{(}mathematician)$
9. Wikipedia: Novikov conjecture
`https://en.wikipedia.org/wiki/Novikov`$_{c}onjecture$
10. Wikipedia: William Thurston
`https://en.wikipedia.org/wiki/William`$_{T}hurston$
11. Wikipedia: John Milnor
`https://en.wikipedia.org/wiki/John`$_{M}ilnor$

Gordon O'Connor: Formatting, Parts 1-5
Email address: `G.OCONNOR18@nuigalway.ie`

Cormac Deignan: Parts 6-9
Email address: `C.DEIGNAN1@nuigalway.ie`

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

TOPOLOGICAL DATA ANALYSIS

DAVID O DEA AND PADRAIG LAFFERTY

(Communicated by Julie Bergner)

ABSTRACT. Data is crucial. It is data that primarily influences decisions. Traditional data analysis techniques have not kept pace with the volume and complexity of modern data sets. A new more powerful approach is required. This has arrived in the form of topological data analysis.

1. **Introduction**

A characteristic of modern society is the ever-increasing volume and complexity of data. Modern datasets can be gigantic and high levels of sophistication are required with regard to analysis and interrogation. Topological data analysis, (TDA) is a suite of approaches that are used for dataset analysis and provide many useful additional insights [2]. Any data scientist who would like a deeper and more complete understanding of data will find TDA to be useful, whether it is being used it its own right or as a means of augmenting other forms of analysis.

2. **The Three Properties of Topological Data Analysis**

The three fundamental properties combine in striking ways to allow one to analyse and understand large and complicated data sets.

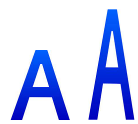
2.1. **Coordinate Invariance.**

The first big concept of topology that is powerful for analysing shape is coordinate invariance. The idea is that topology studies properties of shapes that do not change even as you rotate the shape or change the coordinate system in which you are viewing the shape. In other words, topology studies shape in a coordinate free way, and thus our topological construction only depends on the distance function that specifies the shape. For example, the ellipses below are all considered to be topologically the same irrespective of their positioning in the plane [5].

Received by the editors 06/05/2021.
2020 *Mathematics Subject Classification.* Primary 55N31, Secondary 62R40.

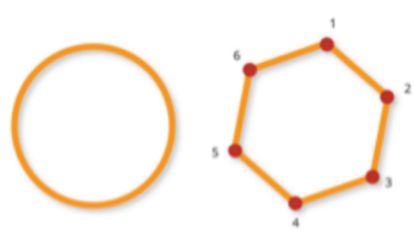
This property is extremely useful in data analysis. Data is often modified using various transformations on data matrix entries. As a result, coordinates are always changing. These transformations are seen in everyday life. Simple transformations involve only translations and scaling and can be seen in converting temperatures from one unit to another. More complex transformations can be seen in three-dimensional rotations and are often useful in better understanding data sets. Whatever the transformation may be, the coordinate invariance still applies. The properties that are studied do not change under such coordinate changes.

2.2. **Deformation Invariance.** Consider a printed letter "A" on a rubber sheet that has been stretched in some directions to indicate deformations.



Although the letter will deform, the two legs and the closed triangle will remain. In a more mathematical context, the invariance property states that a circle, the boundary of a hexagon and an ellipse are all topological similar, because any of these shapes may be obtained by stretching and deforming any of the others. All of these shapes have one thing in common which is that they are all loop. This fundamental property of topology is what allows it to be significantly less sensitive to noise and as a result has the ability to pick out an object's shape despite numerous variations or deformations [3].

2.3. **Compressed Representation.**



Consider the total perimeter of Lough Corrib. A more coarse representation of the lake, such as a polygon, is preferable for examination purposes. Topology is concerned with triangulations or the identification of a shape using a finite combinatorial object known as a simplicial complex or a network [3]. The identification of a circle as having the same shape as a hexagon is a classic example of this type of representation. Only a list of six nodes and six edges, as well as data specifying which nodes correspond to which edges, can be used to describe the hexagon. This is a type of compression in which the number of points is reduced from an infinite number to a limited number. Clearly the circle has has infinitely many points, and infinitely many pairwise distances within. Although the curvature and other information is lost, the fundamental "loop" property is preserved [5].

3. **Relationship between Topology and Datasets**

Topology researchers learn about space by assigning invariant algebraic objects, which can be as simple as integers but are typically more complicated algebraic structures. Persistent homology is the chosen invariant for TDA. The data gathered is typically in the form of an ordered set of N-tuples with attributes such as coordinates and dimensions. When these features are integers, researchers might consider them as definition vectors for European space [1]. To calculate a filtered value for each data point, researchers employ a filter function, which might be a linear projection of the data matrix or the density estimate or the centrality index of the distance matrix. According to their filtered values, the data points are separated into distinct filter value intervals ranging from small to large [1]. In most circumstances, the overlap area between neighbouring filter value intervals is set. That is, the overlap area's points at the same time belong to two intervals. The data is then clustered in each interval separately. If the raw data points in the two classes are identical, an edge must be added between them. To establish equilibrium, a layer of mechanical layout is applied to the above mentioned circle and edge graphics and the final data pattern is created.

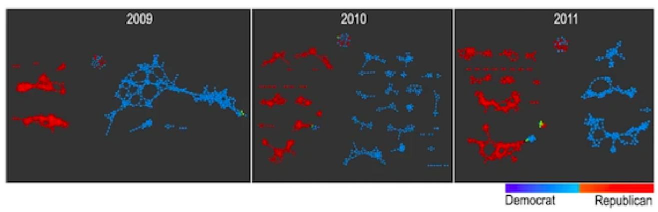
4. **Real Life Applications of TDA**

Data analysis has never been as important as it is to the modern business. Company's want to be able to collect data, analyse it, and present the data in a clear and concise manner. This has opened the door to data visualization - the graphical representation of information and data, and it has become invaluable to businesses [6]. Visualising data can help predict consumer behaviour and understand market trends. In the past, much of this data analysis has been done in the form of histograms, bar charts, scatterplots, and pie charts. However the way forward seems to be in topological data analysis (TDA). The difference between past methods and TDA is that TDA allows you to represent structured and unstructured data through a topological network. This network arranges the data so that nearby points are more similar than distant points. The topological network is similar to a map of a region in that it truly helps to understand the "landscape" of the data [5]. In the following sections, we will examine how these topological networks are connected with real world data, and the widespread role of TDA in influencing business direction.

4.1. **US House Of Representatives Voting Behaviour.** The US political landscape is dominated by two major political parties, the Democrats and the Republicans. Though there are many other smaller parties and individuals with various affiliations, the political system is effectively a duopoly. For most of the time, elected individuals vote along party lines, but a closer analysis of voting patterns reveals the existence of subgroups within each party. Often these subgroups appear to have more in common with the opposing party rather than their own party colleagues.

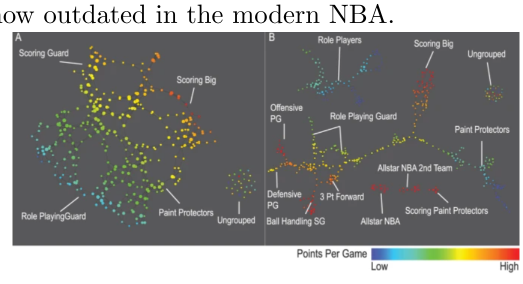
Traditional analysis methods have been shown to have shortcomings in this type of analysis [3]. An application of TDA to a data set comprising 22 years of voting records for members of the US House of Representatives is interesting and can be said to give credence to the claim that there liberal Republicans and conservative democrats exist based on voting behaviour patterns being closely scrutinised. Sometimes these subgroups are temporary in nature and often show large levels of fragmentation. At other times, the subgroups can show high levels of cohesion. The sheer level of detail required to establish any type of pattern or trend was beyond traditional data analysis methodologies. TDA is able to perform the required levels of analysis and generate the various networks that enable data analysts to see subgroups with much greater focus.

In the study period that provided the data set, TDA identified the largest subgroup fragmentation in the 2008-2010 era [3]. This corresponded to major issues around the financial collapse and the advent of Obamacare. More cohesive cross-party subgroup affiliations were obvious at other times and concerning issues like The Credit Cardholders' Bill of Rights, the Highway Trust Bill and Southern Sea Otter Recovery and Research Act. Herein, a Republican subgroup voted like Democrats. This subgroup continued to vote like another subgroup of Democrats on certain issues over a sustained period of time. Like-minded subgroup members included Sherwood Boehlert (R-NY) and Ike Skelton (D-MO). Strong cross-party connections became evident. Such stratification may not have been identifiable without TDA.



In the above plot, the actual topological networks for the members can be seen. The networks are constructed from the voting behaviour of the members of the house. Each node corresponds to a set of members [3]. This way of connecting the data helps paint a picture of how the house is voting, irrespective of party. This way of presenting data truly highlights the cross-party connections shown above.

4.2. **National Basketball Association.** Another application of TDA is to a data set that encodes many aspects of performance among NBA basketball players as a way of identifying more diverse playing styles. More playing styles were identified beyond the traditional five by using rates of points scored, assists, rebounds, steals, blocks, turnovers and personal fouls. The analysis' distance metric and filters were variance normalized Euclidean distance and principal and secondary singular value decomposition (SVD) values, respectively [3]. The positions in basketball are classified into five distinct positions, point guard, shooting guard, small forward, power forward and center. These five positions cover a wide range of players, from those who are short, fast and play outside the paint to those who are tall, slow and play inside the paint. However, classifying these players based on their physical characteristics is now outdated in the modern NBA.



We see that there is a much finer structure in the networks than the five typical positions. Based on the players in game performance statistics, these structures reflect groups of players. On the left side of the main network, there is a finer stratification of guards, the guards are divided into three categories offensive point guards, defensive point guards and ball handling shooting guards [3]. Three smaller structures labeled “All star NBA” and “All star NBA 2nd team” may also be found in the lower middle area of the map. The “All NBA” network is made up of the NBA’s best players and the second team also has superb all around players that are not quite at the level of the “All NBA” players. Within the “All star NBA” group are all star players like Giannis Antetokounmpo, Kevin Durant and Lebron James. The “All star NBA” and “All star NBA 2nd team” networks are clearly isolated from the large network, showing that these players in-game statistics are vastly different. The high resolution network and the lower resolution network were compared to demonstrate the capability of performing multi-resolution analyses simultaneously on the same dataset [3]. The data on the right side demonstrates that at a lower resolution the players are divided into four categories scoring big men, paint protectors, scoring guards and ball handlers. In conclusion, this use of TDA indicates that players should be divided into thirteen positions rather than the traditional five because players are now more versatile in how they play.

4.3. **Ayasdi.** TDA is an area of acute interest to scientists and entrepreneurs worldwide. This is evidenced by both the many data analytical companies that currently exist, and the rate at which new companies are emerging. One such company is Ayasdi, whose mission is to make sense of the world’s complex data. Ayasdi, through TDA, aims to simplify the extraction of intelligence from even the most complex data sets confronting organizations today [4]. Ayasdi combines advanced learning algorithms, abundant compute power and topological summaries that have recently been honed by Stanford computational mathematicians. Ayasdi focus on a wide variety of applications, ranging from the empowerment of personnel to make better business decisions, to assisting the forces of Law and Order in the fight against financial crimes, whether that be fraud, money laundering, embezzlement, etc. Customers include GE, Citigroup and UCSF among others.

References

1. Cheng, L. *The Application of Topological Data Analysis in Practice and Its Effectiveness*, E3S Web of Conferences **214**, 03034 (2020), DOI 10.1051/e3sconf/202021403034
2. Duve, R.(2020) *Intro to Applied Topological Data Analysis.* Medium. Towards Data Science. `https://towardsdatascience.com/intro-to-topological-data-analysis-and-application-to-nlp-training-data-for-financ`
3. Lum, P., Singh, G., Lehman, A. et al. *Extracting insights from the shape of complex data using topology*, Sci Rep **3**, 1236 (2013), DOI 10.1038/srep01236
4. Unknown Author, *Symphony Ayasdi* `https://www.ayasdi.com/`
5. Unknown Author *Topology and Topological Data Analysis* Ayasdi `https://www.ayasdi.com/wp-content/uploads/2015/02/Topology`$_{and}$*T*$opological_{D}ata_{A}nalysis.pdf$
6. Unknown Author, *Why Data Visualization Is Important* Analytiks. `https://analytiks.co/importance-of-data-visualization/`

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

LINDELÖF TOPOLOGICAL SPACES

KATIE O'DOWD AND SINEAD MC MAHON

(Communicated by David Futer)

ABSTRACT. In this paper we discussed Lindelöf topological spaces. We discuss the history of Erst Leonard Lindelöf and his contributions to mathematics and topology. This paper discuss the definition of Lindelöf and what lead him to discover his theorem. We look at his theorem and examples of Lindelöf spaces. We look at the properties of Lindelöf topological spaces.

1. History

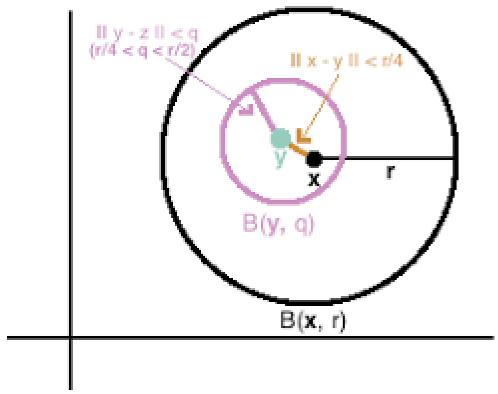
Erst Leonard Lindelöf was born in Helsingfors ,Finland on the 7th of March 1870. Erst Lindelöf's father was a professor in mathematics in Helsingfors , so it was no wonder that he followed in his footsteps as he choose and devoted his life to being a professor in mathematics also.
Lindelöf worked on and contributed to many mathematical theories and solutions that we still use today like differential equations, complex analysis, analytic functions and topology.
Lindelöf encouraged the importance of studying the history of mathematics in his country and was honoured by universities of Uppsala, Oslo, Stockholm, and Helsinki.
Lindelöf gave up researching and devoted hinself to teaching and writing mathematics textbooks such as Differential and integral calculus and their applications which was published in four volumes between 1920 and 1946.

2010 *Mathematics Subject Classification.* Primary .

Famously, Lindelöf spaces are named after him.

2. Definition

A topological space is called a Lindelöf space if each of its open covers has a countable subcover.



3. Lindelöf Theorem

3.1 What led Lindelöf to discover his theorem

E.L. Lindelöf found an important result concerning an m-dimensional, Euclidean space, R^m.
A family F of sets is a base for a topology τ, if and only if, F is a subfamily of τ and for each point x of the space and for each neighbourhood U of x, there is a member V $\in$ F such that x $\in$ V $\subset$ U.
In particular, the space R^m has a countable, open base. That is R^m has a family B of open sets with the following properties.

B1. The family B is countable, i.e. in 1-1 correspondence with the natural numbers.

B2. For each point x in an open subset $G_x \subset R^m$, there is a $B_x \in$ B such that x $\in B_x \subset G_x$

3.2 Lindelöf's Theorem
States: Every open cover of R^m has a countable subcover.
Proof:
If F is any open cover of R^m, then for each x $\in R^m$, there is an open superset G_x from F.
Then there is a $B_x \in$ B such that x $\in B_x \subset G_x$.
Choose only one such G_x for a B_x.
Since B is countable, the family $\{G_x\}$ of supersets of B_xs is a countable subcover of F. $\square (Naimpally and Peters, 2012)$

4. Examples of Lindelöf topological spaces

4.1 Every compact topological space is Lindelöf
Definition:
Let (X,τ) be a topological space. Then an open cover is a set Ui ⊂ X i∈I of open subsets such that their union is all of X

$$\cup \text{ Ui} = \text{X}$$

This is called a finite open cover if I is a finite set.
A topological space is called compact if every open cover has a finite subcover.
Example:
1. For any a < b ∈ R the closed interval

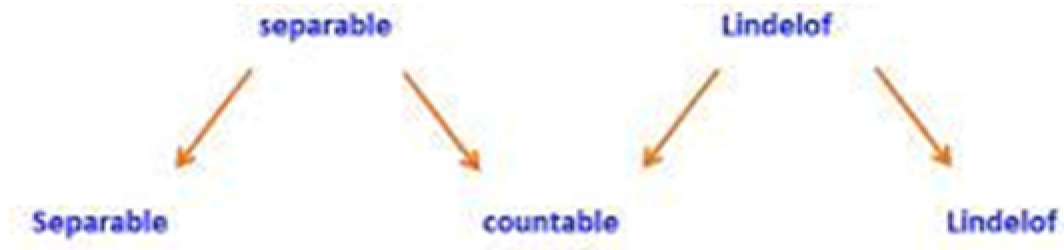
$$[\text{a, b}] \subset \text{R}$$

regarded with its subspace topology of Euclidean space with its metric topology is a compact topological space.
2. From the Heine-Borel theorem every bounded and closed subspace of a Euclidean space is compact.
In particular any set when equipped with the cofinite topology forms a compact space. This space is also T_1 (i.e. singletons are closed), but is not Hausdorff if the underlying set is infinite.
4.2 Every second-countable topological space is Lindelöf
Definition:
A topological space is second-countable if it has a base for its topology consisting of a countable set of subsets.
Example:
1. Let n ∈ N. Consider the Euclidean space R^n with its Euclidean metric topology. Then R^n is second countable.
2. A Hausdorff locally Euclidean space is second-countable precisely it is paracompact and has a countable set of connected components. In this case it is called a topological manifold.
4.3 Every sigma-compact topological space is Lindelöf
Definition:
A topological space is called sigma-compact if it is the union of a countable set of compact subspaces.
Example:
1. Locally compact and second-countable spaces are sigma-compact (authors, 2021).

5. Properties

- Regular Lindelöf spaces are paracompact meaning a topological space X is Lindelöf and regular.
- Second-countable meaning there is a countable base of the topology.
- Metrisable meaning the topology is induced by a metric.
- -locally discrete base: the topology of X is generated by a -locally discrete base.
- Separable meaning there is a countable dense subset.
- Weakly Lindelöf meaning for every open cover there must be a countable subcollection the union of which is dense.
- The product of a Lindelöf space and a compact space is Lindelöf.

Summary: Every countable space is Lindelöf. A Lindelöf space is compact if and only if it is countably compact. Every second-countable space is Lindelöf, but not conversely. For example, there are many compact spaces that are not second countable.



6. References

1. authors, n., 2021. Lindelöf topological space in nLab. [online] Ncatlab.org. Available at: ¡https://ncatlab.org/nlab/show/Lindel
2. Naimpally, S. and Peters, J., 2012. Topology with applications. 1st ed. Singapore: World Scientific Publishing Co. Pte. Ltd., pp.21-22.
3. Ncatlab.org. 2021. Lindelöf topological space in nLab. [online] Available at: ¡https://ncatlab.org/nlab/show/Lindel
4. McCluskey, A. and McMaster, B., 2014. Undergraduate Topology a working text book. 1st ed. Oxford: Oxford University Press, pp.124-150.
5. O' Connor, J. and Robertson, E., 2001. Ernst Lindelöf - Biography. [online] Maths History. Available at: ¡https://mathshistory.st-andrews.ac.uk/Biographies/Lindelof/¿ [Accessed 7 April 2021].
Image 1: En.wikipedia.org. 2021. Ernst Leonard Lindelöf - Wikipedia. [online] Available at: ¡https://en.wikipedia.org/wiki/Ernst$_{L}eonard_{L}indel$
Image 2: Mathonline.wikidot.com. 2021. The Lindelöf Covering Theorem in Euclidean Space - Mathonline. [online] Available at: ¡http://mathonline.wikidot.com/the-lindeloef-covering-theorem-in-euclidean-space¿ [Accessed 7 April 2021].
Image 3: Ma, D., 2021. Weakly Lindelof spaces. [online] Dan Ma's Topology Blog. Available at: ¡https://dantopology.wordpress.com/2014/02/28/weakly-lindelof-spaces/¿ [Accessed 7 April 2021].

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

TOPOLOGISTS WHO HAVE WON THE FIELDS MEDAL

RACHEL TIERNEY AND CHLOE FLOOD

Abstract. The Fields medal is regarded as one of the highest honors a Mathematician can receive. In this article we will go into more detail on what the Fields Medal is and its importance in Mathematics. We will then give a brief account of three of the recipients Jesse Douglas, Klaus Roth and Maryam Mirzakhani.

1. What is the Fields Medal?

The Fields medal can be thought of as the "nobel prize" in relation to Mathematics, it is awarded to two-four Mathematicians under 40 years of age every four years. The Fields medal is to recognise and support younger mathematical researchers. It was first awarded in 1936 to Finnish Mathematician Lars Alfhors and American Mathematician Jesse Douglas. The second time the Fields medal was awarded was in 1950 and since then it has been awarded every four years since then, being awarded to 60 people in total as of 2018. With an exception of one which has a PhD in Physics only people who hold a PhD in Mathematics have won the Fields Medal. Since first being awarded in 1936 there has been only one Female awarded the Fields medal, Maryam Mirzakhani who was awarded it in 2004 who will be one of the three recipients we will cover in this article.(Fields Medal - Wikipedia, 2021)

Figure 1. The Fields Medal

2010 *Mathematics Subject Classification.* Primary .

2. Recipients of the Fields Medal

2.1. **Jesse Douglas.**

Jesse Douglas was born in New York on the 3rd of July 1897. After graduating high school he went to the City College of New York where he won the Belden medal for excellence in mathematics in his first year of study, he was the youngest recipient of this award at the time. He graduated in 1916 and then went on to obtain a PhD in Mathematics in Columbia University in 1920. Jesse Douglas was one of the first recipients of the fields medal in 1936 for solving the problem of Plateau.(Jesse Douglas - Wikipedia, 2021)

Figure 2. Jesse Douglas

Research

Plateau's problem is the problem of finding minimal surface with a given boundary Γ.This problem was first formulated by Lagrane in 1760, but experiments by J.Plateau in 1849 showed that the minimal surface can be obtained using soap films resulted in it being named "Plateau Problem".

In 1931, T.Rado came up with a solution to the Plateau problem for a simply connected surface.After this, Douglas formulated the "Douglas Problem" on the existence in R^n, $n \geq 2$, of a minimal surface having a given topological type and bounded by a given contour Γ consisting of the union of $k \geq 1$ jordan curves $\Gamma 1, ...\Gamma k$.(Plateau's problem - Wikipedia, 2021)

After giving a complete solution to the Plateau Problem,Jesse went on to study the generalisations of it and published papers on these generalisations and in 1943 he was awarded the Bôcher prize by the American Mathematical Society for his memoirs on the Plateau Problem.(O'Connor and Robertson, 2006)

2.2. **Klaus Roth.**

Klaus Roth was born in Germany on the 29th of October 1925 at a young age his family moved to England. He attended the University of Cambridge and University College of London where he received a PhD in 1950 with his dissertation being a proof that almost all Positive Integers are Sums of a Square, a Positive Cube and a Fourth Power. Between his studies at Cambridge and his graduate studies Klaus briefly became a schoolteacher. Klaus taught at University College of London until 1966 when he took a chair at Imperial College London.

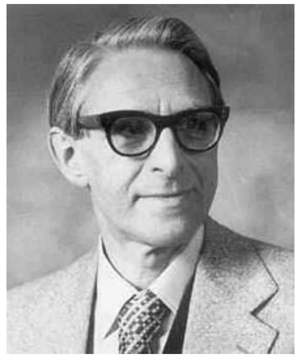


FIGURE 3. Klaus Roth

Research

In 1958 in Edinburgh Klaus Roth won was the first recipient to receive the fields medal in the UK, he was affiliated to the University College of London at the time. Klaus won this award for his research in Diophantine approximation which seeks accurate approximations of irrational numbers by rational numbers. The Thue-Siegel problem is how accurately algebraic numbers can be approximated. This can be measured by an approximation component of a given x which is the largest number e such that x has infinitely many rational approximations p/q with $|x-p/q| < 1/q^e$. A large approximation exponent gives more accurate approximations for x with the smallest approximation exponent being 2. Before Roth's work it was believed that the algebraic numbers could have a larger exponent related to the degree of the defining polynomial. In 1955 Klaus published Roth's Theorem which proves that for irrational algebraic numbers, the approximation exponent is always exactly 2. Klaus also worked on other aspects of mathematics like arithmetic combinatorics, discrepancy and sum of powers. After winning the fields medal, he was elected to the Royal society, he then became an honorary member of the Royal Society of Edinburgh and in 1991 Klaus won the Sylvester Medal for his contributions to number theory and Roth's Theorem. In 2009 a collection of 32 essays on topics related to Roth's research was published, in 2017 the journal Mathematika dedicated an issue to Roth and later the Imperial College Department of Mathematics developed a scholarship in his honour.(Klaus Roth - Wikipedia, 2021)

2.3. Maryam Mirzakhani

.

Maryam was born on the 12th of May 1977 in Iran. Maryam studied at Sharif University of Technology where she received a Batchelor in Science and the went on to receive a PhD from Harvard University. After her studies Maryam was research fellow of the Clay Mathematics Institute and a professor at Princeton University in 2004. She became a professor of mathematics at Stanford university in 2009. In 2005 she was acknowledged as one of the "Brilliant 10" in Popular Science's fourth annual for innovation in her field. On the 13th of August 2014 Maryam became the first Iranian and the first female to receive the Fields Medal. Maryam received the award due to her work in the dynamics and geometry of Riemann surfaces and their moduli spaces.

FIGURE 4. Maryam Mirzakhani

Research

In her thesis, Maryam found a formula for the volume of the moduli space of bordered Reimann surfaces of genus g with n geodesic boundary components which resulted in solving the problem of counting simple closed geodesics on hyperbolic Riemann surfaces. In simple language a geodesic is a curve that there is no slight deformation that can shorten it, it is closed if it closes into a loop and simple means it does not cross itself. Maryam's thesis in 2004 revealed the number of simple closed geodesics of length less than L is asymptotic to cL^{6g-6} with g meaning the number of holes and c a constant arising from hyperbolic structure. Later on she proved the long-standing conjecture that William Thurston's earthquake flow on Teichmüller space is ergodic. Maryam looked at an earthquake as a limit of simple earthquakes with infinite geodesics which you can place a measure on. In 2014 alongside Alex Eskin and help form Amir Mohammadi Maryam proved complex geodesics and their closures which are objects defined in terms of polynomials which often have rigid properties in moduli space are surprisingly regular.(Maryam Mirzakhani - Wikipedia, 2021)

3. Conclusion

Having researched our three chosen recipients of the Fields Medal we have seen how much work and dedication they spent on their research that ended up in them receiving one of the most prestigious awards in Mathematics.

4. References

[1] En.wikipedia.org. 2021. Fields Medal - Wikipedia. [online] Available at: ¡https://en.wikipedia.org/wiki/Fields$_M$*edal* > [*Accessed*2*June*2021].
[2]*En.wikipedia.org.*2021.*JesseDouglas−Wikipedia.*[*online*]*Availableat* :< *https* : *//en.wikipedia.org/wiki/Jesse*$_D$*ouglas* > [*Accessed*2*June*2021].
[3]*O′Connor, J.andRobertson, E.*, 2006.*JesseDouglas−Biography.*[*online*]*MathsHistory.Availableat* :< *https* : *//mathshistory.st−andrews.ac.uk/Biographies/Douglas/* > [*Accessed*2*June*2021].
[4]*En.wikipedia.org.*2021.*Plateau′sproblem−Wikipedia.*[*online*]*Availableat* :< *https* : *//en.wikipedia.org/wiki/Plateau*27*s*$_p$*roblem* > [*Accessed*2*June*2021].
[5]*En.wikipedia.org.*2021.*KlausRoth* − *Wikipedia.*[*online*]*Availableat* :< *https* : *//en.wikipedia.org/wiki/Klaus*$_R$*oth* > [*Accessed*2*June*2021].
[6]*En.wikipedia.org.*2021.*MaryamMirzakhani−Wikipedia.*[*online*]*Availableat* :< *https* : *//en.wikipedia.org/wiki/Maryam*$_M$*irzakhani* > [*Accessed*2*June*2021].

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

TOPOLOGICAL APPLICATIONS IN FINANCIAL MATHEMATICS

LISE WALL, EMMA MEANEY, CATHAL BOYCE

(Communicated by Julie Bergner)

ABSTRACT. The study of Topology provides Financial Mathematicians with new and innovative tools to study financial markets. These topological methods can compliment and improve on traditional analytical tools such as statistical regression and linear algebra. Topological Data Analysis (TDA) is a method that can be used to study multivariate data sets. The Mapper algorithm is a TDA tool which can be applied to firm financial data to obtain stock valuation while Persistent Homology can be used to detect Stock Market crashes. Stochastic Flow Diagrams are topological representations of systems of complex equations which can be used to model financial systems such as markets, which we discuss below.

1. Financial Data Analysis using the Mapper Algorithm

1.1. **An Overview:** The mapper algorithm is a Topological Data Analysis tool that can be used to provide insights into financial data sets. It provides a way to visualise the intrinsic shape of high dimensional data in a single graph. This allows data scientists to obtain a coarse overview of the data set and can point to areas of interest. Fundamental Analysis is a method of determining the intrinsic value and potential future value of an asset by analysing external data sources such as firm financial statements and macroeconomic conditions. A variation of the original Mapper Algorithm can be applied to firm financial data, and stock returns in order to identify anomalies in asset pricing which can be exploited for financial gain.

1.2. **Outline of the Mapper Algorithm:** The mapper algorithm starts with data points in high dimensional space. The data points are mapped to R using a filter function. A covering is found for the given data by dividing this range into a set of smaller overlapping intervals. This provides 2 important parameters which control resolution: the length of the intervals and the overlap between them. Each interval in the cover is clustered. An output graph is produced consisting of nodes and edges. Nodes represent clusters of data points while edges represent non-empty intersections between pairs of clusters. In other words edges connect clusters which have data points in common.[1]

Received by the editors 07/05/21.
2010 *Mathematics Subject Classification.* Primary 55S37; Secondary 55N31.

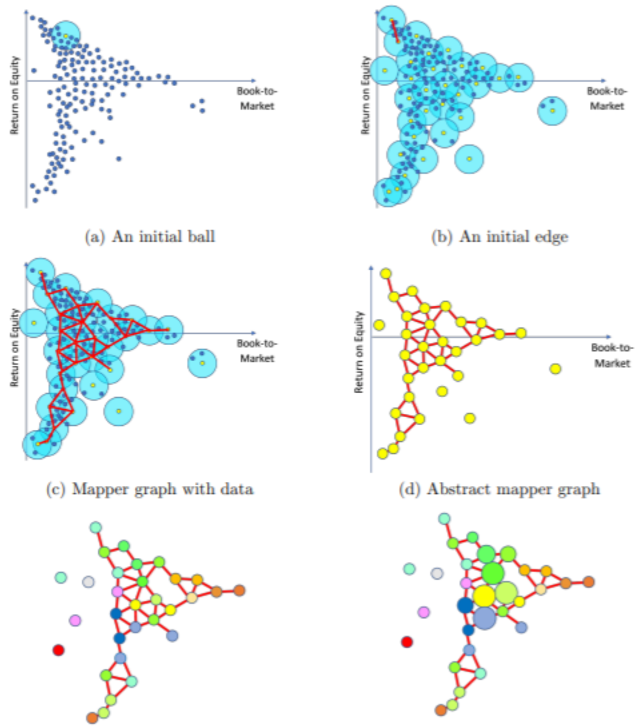
1.3. **The Link between Financial Ratios and Stock Market Returns:** Extensive research has been done on the relationship between the financial ratios of a firm and the movement of its stock price. Traditionally,financial variables x which are determinants of stock market returns are identified by using linear regression on x along with variables that are known to influence returns(controls) to analyse the significance of relationships between x and the returns. If the relationship is significant x can be included in the stock return model. A major weakness of this method is that it assumes linear dependence of returns from data and only accounts for the relationship between returns and individual variables rather than their combinations. This can be overcome by Topological Data Analysis which does not rely on functional form to map characteristics to outcome.
Thousands of potential fundamental signals have emerged from financial statements. These can be categorised into Profitability, Liquidity, Activity, Debt and , Valuation ratios. For example the Return on Equity and Return on Assets are profitability metrics, while the book-to-market ratio, the Price/Earnings ratio and the Cashflow-to-price ratio are valuation metrics. A combination of profitability and valuation ratios such as the book-to-market ratio and Return on Equity are often used together to gain a more comprehensive picture of a stock's intrinsic value.

1.4. **Using the Mapper Algorithm to identify Anomalies:** The Ball Mapper Algorithm is a variation of the original mapper algorithm. The algorithm produces a cover of the space X with a collection of overlapping sets Ci. It is assumed that the points in each set are geometrically close. A set Ci is obtained by selecting a point x at random from the data, and including all the points a fixed distance ϵ from x in Ci. The choice of the radius ϵ is important as overly large values can neglect some of the finer detail while overly small values can create maps which are difficult to interpret. The sets Ci form vertices of a graph, and edges are formed between sets with non-empty intersections. The weight of a vertex is its cardinality while the weight of an edge is the number of elements in the intersection of the two vertices it connects. This is visualised in the graph by varying sizes of vertices [2].

Colourisation of the graph provides further insight. A function f maps the points in X to the real line R. A new function f' is defined which produces an average value of f for all values in each set Ci. The vertices are coloured according to the resulting values of f'. Variations in colour across vertices can point to areas of interest. Differences in colour between connected vertices are indicative of the significant effects of small variations in combinations of variables. This allows analysts to identify non-linearities and patterns within the data that linear modelling may overlook [2].
The colourisation of balls in the plots (e) and (f) below represent the expected returns with colours set on a spectrum from red to yellow to green to blue to mauve. The graph shows that the highest returns lie in the region of low book-to-market and return on equity near to zero.The effects of unilaterally changing one variable are evident. Increasing book-to-market but not return on equity causes returns to decrease but not as starkly as if return-on-equity were increased with book-to-market constant.This is consistent with some results obtained from traditional regression methods however as the analysis is expanded to many more financial variables such as profitability, earnings-to-price and dividend yield, Topological data analysis can be more suitable in identifying non-linear patterns. For example, some non-linearity

is evident when the ball mapper algorithm is applied to the effect of profitability and firm size on returns. A combination of high profitability and large firm size is linked to higher returns however a nonlinear pattern arises which is not consistent with traditional models [2].

Figure 2: Construction of the TDA Ball Mapper Graph



(a) An initial ball (b) An initial edge

(c) Mapper graph with data (d) Abstract mapper graph

(e) Colouration of plot (f) Ball sizes

1.5. **Conclusion:** Topological Data Analysis, and in particular the Ball Mapper Algorithm can be used to understand the relationships between fundamental signals arising from firm financial information and stock market returns. This method can complement and improve on the statistical regression methods that are used traditionally. Benefits arise from the ability of TDA to identify non-linearities and patterns in relationships between financial variables and returns. It also presents analysts with a useful visualisation method for high dimensional financial data. This can be used to gauge particular areas of interest for further research.

2. Stochastic Flow Diagrams and Economic Shocks

2.1. **An Overview:** Markets are interdependent price discovery mechanisms, where prices are mutually interrelated. The occurrence of a demand or supply shock in any particular market has the ability influence prices across all other markets. Price "stickiness" refers to the inability of prices to reflect new information immediately, resulting in a lead-lag affect.

Traditionally, linear algebra and stochastic calculus are used to analyze the financial system. However, these methods do not capture all relevant information within the system. For example, they provide little information about the number of cycles and pathways. Consider a topological study of the complex, dynamic financial system. These topological relations represent market arrangement and interconnectedness rather than geographical distance. These representations are called stochastic flow diagrams (SFDs).

2.2. **What are SFDs?:** SFDs are used to monitor the flow of capital allocations. They are graphical representations of a Time Series complex dynamic network, consisting of a series of nodes (which represent market participants) and edges (connections between nodes which exist when the correlation coefficient of a pair of nodes exceeds a given significance level). SFDs are directed graphs (diagraphs) whose direction is determined by lead-lag conditions and whose connections are weighted based on Time Series parameter estimates. Some nodes are called "lagged" nodes based on the length of time it takes them to respond to a shock. The Time Series element and lagged nodes illustrate the possibility of trends,lead-lag effects, momentum and crashes within the market. [3]

2.3. **SFD Construction:** For the purpose of this article, we will consider the most basic form of SFD which is generated using autoregressive (AR) processes. AR provides the basic building blocks for the construction of more sophisticated SFDs. SFD variations are generated by increasing the number of lagged and current nodes.

Let $\{y_t\}$ be a collection of real random variables whose value is determined at time t. AR models are recursive models of random variables. In it's simplest form, AR models consist of a single lagged equation (AR(1)),

$$y_t = \text{c} + \varphi_1 y_{t-1} + \varepsilon_t \qquad (1)$$

where φ_1 is the real-valued scale factor for the lagged observation, c is a constant and ε_t is residual standard error. Assume c=0. The information in equation (1) can be represented in a weighted diagraph D= $(V_D,\, A_D,\, \omega_D)$ where,

- $V_D = \{\upsilon_0,\, \upsilon_1\}$
- $A_D = \{o_0,\, a_1\}$
- $\psi_D[o_0] = (\upsilon_0,\, \upsilon_1)$
- $\psi_D[a_1] = (\upsilon_1,\, \upsilon_0)$
- $\omega_D[o_0] = 1$
- $\omega_D[a_1] = \varphi_1$

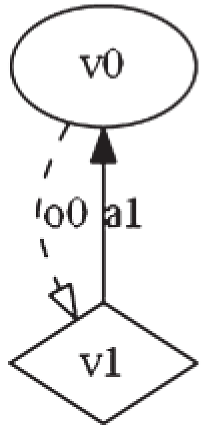
Each node (also called a vertex) represents a particular state of a random variable. The vertex υ_0 represents the value of y_t and υ_1 represents the value of y_{t-1}. These vertices are associated with the time structure of the model and can be described as *lagged* or *current vertex*. Vertices are connected by edges (also called arcs) which can be *outbound* or *inbound*. [4]

Definition (Lagged and Current Vertices): the value of a lagged vertex is υ_k. It has been lagged k times and represents the state of a random variable at y_{t-k}, where $k \geq 0$. When $k = 0$, υ_0 is a current vertex. In the diagram, a current vertex is represented by an ellipse and a lagged vertex is represented by a rhombus.

Definition (Outbound Arc): for vertices υ_k, υ_{k+1} $\in$V$_D$, an arc o_0 $\in$A$_D$ is outbound if it joins the vertices and $\omega_D[o_k] = y_{t-k}$. Arc o_0 is outbound because flow is directed from υ_0 to υ_1, with lag of weight of 1.

Definition (Inbound Arc): for vertices υ_k, u_0 $\in$V$_D$, where υ_k relates to y_{t-k}, u_0 relates to x_t and x_t, y_t distinct.The arc a_k $\in$A$_D$ is inbound if it joins the vertices, with $\omega_D[a_k] = \varphi_k y_{t-k}$ and $k \geq 0$. Inbound arcs direct flow from lagged vertex υ_1 to current vertex υ_0.

The diagraph below illustrates Eq. (1) and it's single lag structure.



In general, a system with p time lags can be expressed as an AR(p) equation,

$$y_t = \sum_{k=1}^{p} \varphi_k y_{t-k} + \varepsilon_t \quad (2)$$

where φ_k is the real-valued scale factor for the k^{th} lagged observation, c is a constant and ε_t is residual standard error [4]. The information in equation (2) can be represented in a weighted diagraph D= (V_D, A_D, ω_D) where,

- $V_D = \{\upsilon_k\}$
- $A_D = \{a_{k+1}\} \cup o_k$
- $\psi_D[o_k] = (\upsilon_k, \upsilon_{k+1})$
- $\psi_D[a_{k+1}] = (\upsilon_{k+1}, \upsilon_0)$
- $\omega_D[o_k] = 1$
- $\omega_D[a_{k+1}] = \varphi_{k+1}$

2.4. **Economic Shocks and SFDs:** SFDs are composed of a network interrelated random variables. Economic shocks can be unexpected and have the ability to influence these variables. Shocks dissipate within the system (using the inbound arcs). A shock can lead to a chain reaction within the system, resulting in a series of complicated reactions which further upset the system. Shocks can generate a stable, steady or explosive state.[4]

2.4.1. *Stable States:* Consider a system with two equations. For example, consider stocks and bonds within the financial system. Suppose stocks are positively correlated with their own lagged value and negatively correlated with the lagged value of bonds (vice-versa for bond relations). A stable system will always absorb the shock and stabilize, irrespective of shock significance. Overtime, rate of change within the system will converge to zero and the system will reach a new equilibrium.

2.4.2. *Steady States:* In a steady state system, the system does not self regulate after a shock. There is a persistent rate of change which endures until another shock occurs and offsets it. Complex interactions between random variables accumulate in perpetuity.

2.4.3. *Explosive State:* In an explosive state system, the occurrence of a shock sends the system into chaos. The rate of change is inflated after each time period. Each state becomes increasingly unsettled until the system eventually crashes.

2.5. **Conclusion:** Stochastic flow diagrams are topological representations of systems of complex equations. SFDs graphically describe lengthy formulae. SFDs are Time Series models and because of this they can incorporate lead-lag effects. Using SFDs we can examine the state of a system at a given time. Furthermore, we can use SFDs to investigate whether a system is stable, steady or explosive subject to an economic shock. SFDs can be used to complement Linear Algebra and Stochastic Calculus to obtain a more thorough understanding of the complex financial system.

3. Predicting the unpredictable: Detecting stock market crashes with topological data analysis

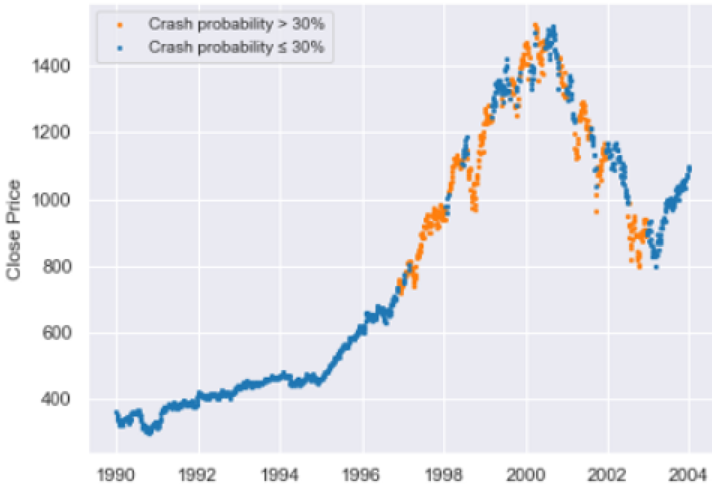
3.1. **An Overview:** Topology is so useful in data analysis because it can take extremely complex and advanced shapes or data points and reduce them to their intrinsic properties. Topology can provide a powerful method to abstract structure in complex data.

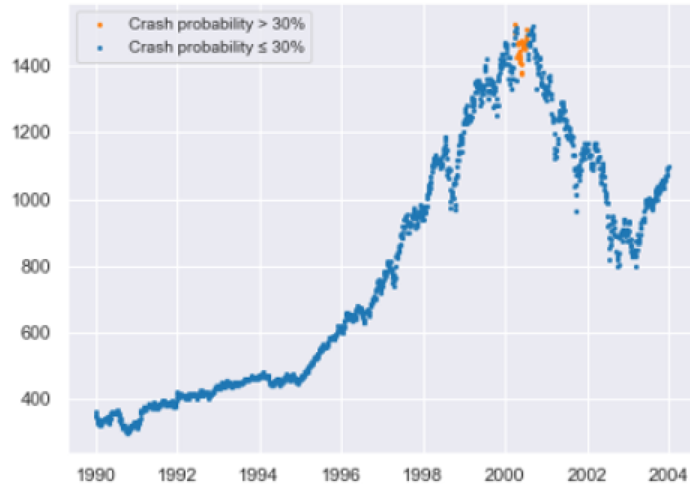
Stock market crashes are rapid drops in asset prices. The price drop is caused by massive selling of assets, which is an attempt to close positions before prices decrease even further.

These crashes and other unexpected events are difficult to predict using conventional methods. As a system reaches a breaking point, the data points representing the system begin to form shapes that change its overall structure. By keeping a close eye on a system's data point clouds, one can identify the system's normal state and thereby also detect when an abrupt change is about to occur such as a phase transition between a solid and liquid.

3.2. **A look at past data:** Data shows that financial crashes are preceeded by periods of abnormal and increased oscillations in the prices of assets - this translates into an abnormal change in the geometric structure of the time series.

The below figure shows the S&P 500 index prices from 1980 to present. Given that market crashes represent a sudden decline of stock prices, one simple approach to detect these changes involves tracking the first derivative of average price values over a rolling window. This can also detect when crashes are about to occur, but is widely inaccurate and will produce a large amount of "false negatives".



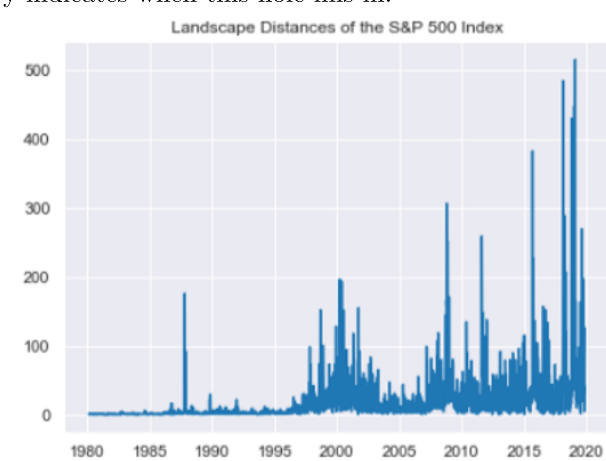


The figure on the left shows the simple approach relating to first derivatives of stock prices while the figure on the left shows the Topological Data Analysis approach. There is clearly a large amount of noise in the first figure, and the detector is not of much use. [5]

3.3. **Topological Data Analysis:** A simple way to think of Topological Data Analysis (TDA) is as a means of extracting information and features from data which can be used for modelling. A starting point for TDA can be to generate a simplicial complex from a point cloud, and to represent a time series as a point cloud, i.e. a set of vectors in Euclidean space of any dimension. [5]

The question at this stage is how this information can be used. This is where "Persistent Homology" comes in, which looks for topological features in a simplicial complex that persists over a range of parameter values.

The point clouds have assosciated persistence diagrams, which is a diagram which represents points which indicate a topological feature which begins at scale x and persists until scale y. For example, a 0-dimensional topological feature is a connected component or cluster, a 1- dimensional topological feature is a hole, and so on. The scale y indicates when this hole fills in.



Why are these useful? Given point clouds and their persistence diagrams, we can calculate a variety of distance metrics which is at the very foundation of measuring the beginning of potential dips or crashes in the market. A persistence diagram can be mapped into a function space by a persistence landscape, which is often taken to be a Banach space. Furthermore, the mapping from persistence diagrams to persistence landscapes is stable and invertible.

3.4. **A Topological Indicator:** Using the landscape distance as a topological feature yields interesting results. As shown below, we can see the resulting detection of the stock market crashes for the dot-com bubble and the 2008 financial crisis. Compared to the original image of the first derivative model, there is clearly a massive improvement in the reduction of noise and improvement in reliability and accuracy using this topological approach.

3.5. **In Conclusion:** Periods of high volatility often preceed a crash, as described earlier. This volatility produces geometric signatures which can be detected, recognised and built into any AI system using TDA as its fundamental.

4. Bibliography

[1]Singh, G., M´emoli, F., Carlsson, G. E. (2007). Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition. Retrieved 05/05/2021 from https://diglib.eg.org/bitstream/handle/10.2312/SPBG.SPBG07.091-100/091-100.pdf?sequence=1isAllowed=y

[2]Dlotko, P., Qiu, W., Rudkin, S. (2019). Financial ratios and stock returns reappraised through a topological data analysis lens. Retrieved 05/05/2021 from https://arxiv.org/pdf/1911.10297.pdf

[3] N.J. Calkin and M. López de Prado, *Algorithmic Finance*, The topology of macro financial flows: An application of stochastic flow diagrams, **3** (2014), no.2, 43-85, DOI 10.3233/AF-140033.

[4] N.J. Calkin and M. López de Prado, *Algorithmic Finance*, Stochastic flow diagrams, **3** (2014), no.1, 21-42, DOI 10.3233/AF-140032.

[5]Brouet, A. (2021). Topological data analysis can help predict stock-market crashes from https://actu.epfl.ch/news/topological-data-analysis-can-help-predict-stock-m/