

MA342: Tutorial Problems 2018-19

Tutorials: Wednesday, 2-3pm, Venue = AC201
Wednesday, 6-7pm, Venue = AM150

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PROBLEMS

1 Euler characteristics

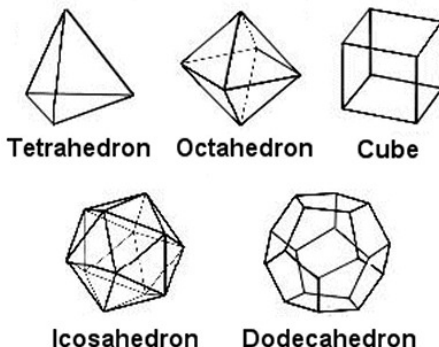
1. Draw a graph on a sphere \mathbb{S}^2



in such a way that if two edges intersect then they intersect in a vertex of the graph. Determine the number of vertices V , edges E and faces F for your graph. Then compute the Euler characteristic $\chi(\mathbb{S}^2) = V - E + F$.

2. Prove that the value of the Euler characteristic $\chi(\mathbb{S}^2) = V - E + F$ in Problem 1 does not depend on your particular choice of graph on the sphere. [See Lecture 1.]
3. A *platonic solid* is a 3-dimensional convex object whose surface is the union of a finite number of polygonal planar faces satisfying:
 - (a) all faces are congruent to some fixed regular p -gon;
 - (b) the intersection of two faces is either empty or a common edge of the two faces or a common vertex of the two faces;
 - (c) the same number of faces, q , meet at each vertex.

Five platonic solids are shown in the following figure.



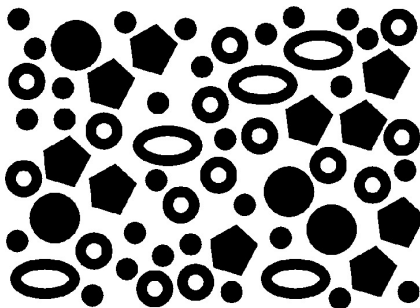
Use the Euler characteristic to prove

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}, \quad e \geq 0$$

for any platonic solid.

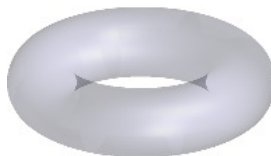
4. Deduce from Problem 3 that there are only five platonic solids.

5. The digital image



represents a region $X \subseteq \mathbb{R}^2$ formed as a union of various unit squares $[m, m + 1] \times [n, n + 1]$ for various integers m, n . Determine $\chi(X)$.

6. Draw a graph on a torus \mathbb{T}



in such a way that

- (a) if two edges of the graph intersect then they intersect in a vertex of the graph;
- (b) each resulting face on the torus is a curvilinear disk (*i.e.* a “continuous deformation” of some planar polygonal disk).

Determine the number of vertices V , edges E and faces F for your graph. Then compute the Euler characteristic $\chi(\mathbb{T}) = V - E + F$. [The term “continuous deformation” will be made precise later in the course: it is just a *homeomorphism*.]

- 7. Prove that the value of the Euler characteristic $\chi(\mathbb{T}) = V - E + F$ in Problem 6 does not depend on your particular choice of graph on the torus. [Hint: The torus \mathbb{T} can be constructed from a rectangular sheet of paper by identifying/gluing opposite sides of the sheet. We know that the Euler characteristic of a solid plane recangle is 1.]
- 8. A *polygonal surface* is a union of curvilinear polygonal disks such that, if two polygonal disks intersect, then their intersection is a union of edges and/or vertices of the disks. The polygonal disks are called *faces*. The soccer ball is an example of a

polygonal surface with pentagonal and hexagonal faces. For any polygonal surface X we define the *Euler characteristic* $\chi(X) = V - E + F$ where V is the number of vertices on X , E the number of edges and F the number of faces.

Suppose that X is a polygonal surface. Let $A, B \subseteq X$ be subsets of X each arising as a union of faces. Prove that

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

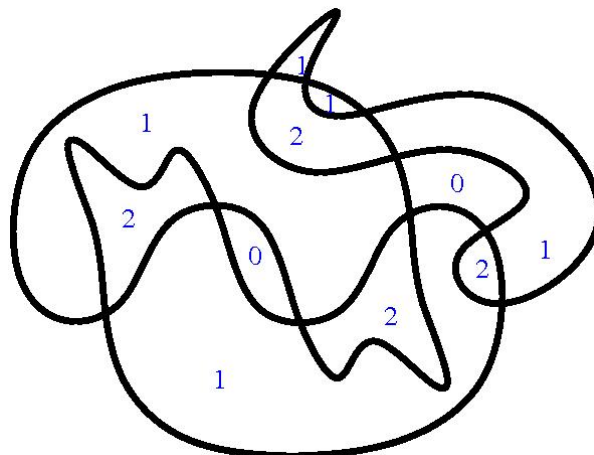
[See Lecture 2.]

- Use the formula $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ to determine the Euler characteristic of a double torus surface.



2 Euler Integration

- The following picture shows the boundaries of several regions $U_1, \dots, U_t \subseteq \mathbb{R}^2$ of common Euler characteristic $\chi(U_i) = 1$.



No two boundaries are tangential at any point. The numbers in the interiors of the regions and their intersections represent the weight function $\omega: X \rightarrow \mathbb{N}$ where $X = U_1 \cup U_2 \cup \dots \cup U_t$ and $w(x) = |\{i : x \in U_i\}|$. Evaluate the Euler integral

$$\int_X \omega d\chi$$

and then determine the number of regions t .

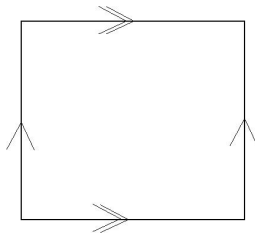
- Let $X \subseteq \mathbb{R}^2$ be a region arising as the union of subregions $U_1, U_2, \dots, U_t \subseteq X$ of common Euler characteristic $\chi(U_i) = C$. Let $\omega: X \rightarrow \mathbb{N}$ be the weight function given by $w(x) = |\{i : x \in U_i\}|$. Prove that

$$t = \frac{1}{C} \int_X \omega d\chi .$$

[See Lecture 4.]

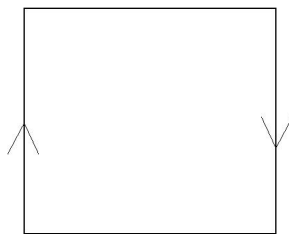
3 Möbius strips, Klein bottles ...

- The torus \mathbb{T} is obtained from the unit square $[0, 1] \times [0, 1]$ by making the identifications $(x, 0) = (x, 1)$ and $(0, y) = (1, y)$ for $x, y \in [0, 1]$.



Is it true that any loop in \mathbb{T} that has no self intersections cuts \mathbb{T} into two components? If not, then exhibit a loop that does not cut \mathbb{T} into two components. [See Lecture 4]

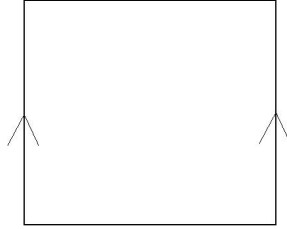
- The Möbius strip \mathbb{M} is obtained from the unit square $[0, 1] \times [0, 1]$ by making the identifications $(0, y) = (1, 1 - y)$ for $y \in [0, 1]$.



Is it true that any loop in \mathbb{M} that has no self intersections cuts \mathbb{M} into two components? If not, then exhibit a loop that does not cut \mathbb{M} into two components. [See Lecture 4]

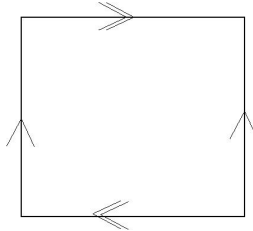
- The cylinder X is obtained from the unit square $[0, 1] \times [0, 1]$ by making the identi-

fications $(0, y) = (1, y)$ for $y \in [0, 1]$.



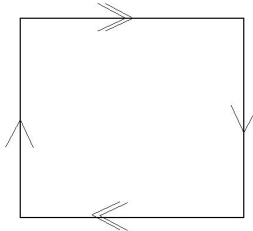
Is it true that any loop in X that has no self intersections cuts X into two components? If not, then exhibit a loop that does not cut X into two components.

4. The Klein bottle \mathbb{K} is obtained from the unit square $[0, 1] \times [0, 1]$ by making the identifications $(x, 0) = (1 - x, 1)$ and $(0, y) = (1, y)$ for $x, y \in [0, 1]$.



Is it true that any loop in \mathbb{K} that has no self intersections cuts \mathbb{K} into two components? If not, then exhibit a loop that does not cut \mathbb{K} into two components.

5. The projective plane \mathbb{P} is obtained from the unit square $[0, 1] \times [0, 1]$ by making the identifications $(x, 0) = (1 - x, 1)$ and $(0, y) = (1, 1 - y)$ for $x, y \in [0, 1]$.



Is it true that any loop in \mathbb{P} that has no self intersections cuts \mathbb{P} into two components? If not, then exhibit a loop that does not cut \mathbb{P} into two components.

4 Subsets of Euclidean space

1. Exhibit a collection of open subsets of the plane \mathbb{E}^2 whose intersection is not open. [Lecture 5]
2. Let $X \subset \mathbb{E}^2$ be the set of those points in the plane that have at least one rational coordinate. Is X an open subset of \mathbb{E}^2 ? Is X a connected subset of \mathbb{E}^2 ? Justify your answers.

3. Let

$$\begin{aligned} Y &= \{(0, y) \in \mathbb{E}^2 : -1 < y < 1\}, \\ Z &= \{(x, \sin(1/x)) \in \mathbb{E}^2 : 0 < x \leq 1\}, \\ X &= Y \cup Z. \end{aligned}$$

Is X an open subset of \mathbb{E}^2 ? Is X a connected subset of \mathbb{E}^2 ? Justify your answers.

5 Topological spaces

- For each of the following sets X and collections T of open subsets decide if the pair X, T satisfies the axioms of a topological space. If it does, determine whether X is connected. If it is not a topological space then explain which axioms fail.
 - $X = \mathbb{R}^n$ and the subset $U \subset X$ is open if, for any $x \in U$, there is a real $\epsilon > 0$ such that the open Euclidean ball $B^n(x, \epsilon)$ of radius ϵ and centred at x is contained in U .
 - $X = \mathbb{R}^n$ and the subset $U \subset X$ is open if, for any $x \in X \setminus U$, there is a real $\epsilon > 0$ such that the open Euclidean ball $B^n(x, \epsilon)$ of radius ϵ and centred at x is contained in the complement $X \setminus U$.
 - $X = \mathbb{R}^n$ and every subset $U \subset X$ is open.
 - $X = \mathbb{R}^n$ and the only open subsets are X and the empty set \emptyset .
 - $X = \mathbb{Z}$ and a subset $U \subset \mathbb{Z}$ is open if and only if its complement $\mathbb{Z} \setminus U$ is finite or $U = \emptyset$.
 - $X = \mathbb{Z}$ and a subset $U \subset \mathbb{Z}$ is open if and only if U is finite or $U = \mathbb{Z}$.
 - $X = \mathbb{R}^n$ and a subset $U \subset \mathbb{R}^n$ is open if and only if it is a vector subspace of \mathbb{R}^n . Here \mathbb{R}^n has the standard addition and scalar multiplication.
 - $X = \{1, 2, 3, 4\}$ and $T = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$.
 - $X = \mathbb{Z}$ and a subset $U \subset \mathbb{Z}$ is open if and only if each of its elements is even.

6 Subspaces

- Given a topological space X , define what it means for a subset $Y \subseteq X$ to be a *subspace*.
- For each of the following topological spaces X and subspaces $Y \subseteq X$ describe the connected components of Y .
 - $X = \mathbb{E}^2, Y = \{(x, y) \in \mathbb{E}^2 : x^2 + y^2 \neq 1\}$.
 - $X = \mathbb{E}^1, Y = \mathbb{Q}$.

- (c) $X = \mathbb{E}^2, Y = \{(x, y) \in \mathbb{E}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$.
 (d) $X = \mathbb{E}^2, Y = \{(x, y) \in \mathbb{E}^2 : x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$.

3. The table

	H	M	R	C	W
H	0	11	10	14	22
M	11	0	3	13	21
R	10	3	0	12	20
C	14	13	12	0	16
W	22	21	20	16	0

gives distances between the species Human, Mouse, Rat, Cat, Whale. For $\epsilon > 0$ let G_ϵ denote the graph with vertices H, M, R, C, W and with an edge between vertices X and Y if $dist(X, Y) \leq \epsilon$.

- (a) Sketch the graphs G_4, G_{10}, G_{16} .
 (b) Explain how one can view the graphs G_ϵ as subspaces of \mathbb{E}^5 .
 (c) Draw the dendrogram that describes the inclusion relationships between the connected components of the subspaces $G_0, G_2, G_4, \dots, G_{18}, G_{20}$.

7 Some useful jargon

- Let X be a topological space and let $W \subseteq X$ be some subset.
 - The subset W is said to be *closed* if the complement $X \setminus W$ is an open subset of X .
 - A point $x \in X$ is said to be a *limit point* of W if every open set $U \subset X$ containing x has non-empty intersection with W .
 - The union of W and all its limit points in X is said to be the *closure* of W . The closure is denoted by \overline{W} .
 - Prove that the closure \overline{W} is a closed subset of X .
 - Suppose that $W \subseteq Z$ where Z is a closed subset of X . Prove that $\overline{W} \subseteq Z$.
 - Prove that \overline{W} is equal to the intersection of all closed subsets of X containing W .
 - Prove that a subset W is closed if and only if $W = \overline{W}$.
- Find a family of closed subsets of the real line whose union is not closed.
- Describe the closure of the subspace $W = \{(1/n) \sin(n) : n = 1, 2, \dots\}$ of the real line.
- Let Y be a subspace of X . Show that if A is closed in Y and if Y is closed in X then A is closed in X .

8 Continuity

1. Give the definition of a continuous function between topological spaces.
2. Let \mathbb{R} denote the real line with its usual topology. Is the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2, & x \leq 1, \\ x^2 + 2, & x > 1 \end{cases}$$

continuous? Justify your answer.

3. Let \mathbb{R} denote the real line with its usual topology. Let $X = \mathbb{R} \setminus \{1\}$ be given the subspace topology. Is the function $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R} \setminus \{1\}$ given by

$$f(x) = \begin{cases} x^2, & x < 1, \\ x^2 + 2, & x > 1 \end{cases}$$

continuous? Justify your answer.

4. Suppose that $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are continuous functions. Prove that the composite $gf: X \rightarrow Z, x \mapsto g(f(x))$ is continuous.
5. Suppose that $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are homeomorphisms. Prove that the composite $gf: X \rightarrow Z, x \mapsto g(f(x))$ is a homeomorphism.
6. Consider the set $X = \{a, b, c\}$ endowed with the topology $T_X = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$, and the set $Y = \{a, b, c, d\}$ endowed with the topology $T_Y = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$. Is the function $f: X \rightarrow Y, a \mapsto a, b \mapsto b, c \mapsto c$ continuous?
7. Let \mathbb{Z} denote the integers endowed with the cofinite topology. Exhibit an example of a function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ which is not continuous.

The first in-class test will consist of a few of the above questions.

4. Prove that the unit circle $S^1 = \{(x, y) \in \mathbb{E}^2 : x^2 + y^2 = 1\}$ is homeomorphic to the square $Y = \{(x, y) \in \mathbb{E}^2 : -2 \leq x, y \leq 2, \text{ and either } x \in \{-2, 2\} \text{ or } y \in \{-2, 2\}\}$.
5. Let Δ denote an equilateral triangular region in the plane \mathbb{E}^2 . Describe the construction of a continuous surjective function $f: [0, 1] \rightarrow \Delta$. (You are *not* asked to prove any convergence nor to prove surjectivity.)
6. Prove that if X and Y are homeomorphic then X is connected if and only if Y is connected.
7. Prove that $(0, 1)$ is homeomorphic to \mathbb{E} .
8. Prove that \mathbb{E} is not homeomorphic to \mathbb{E}^2 .
9. Prove that \mathbb{E} is not homeomorphic to the space $Y = \{(x, y) \in \mathbb{E}^2 : x = 0 \text{ or } y = 0\}$. (Here Y is the union of the x -axis and y -axis.)

9 Compactness

1. Explain why \mathbb{E}^2 is not compact.
2. Prove that the interval $[0, 1]$ is compact.
3. Prove that if X is compact and $f: X \rightarrow Y$ is a continuous map then the image of f is compact.
4. Prove that if X is compact and Y is homeomorphic to X then Y is compact.
5. Determine the accumulation points of the subset $A = \{1/n\}_{n=1,2,3,\dots}$ of \mathbb{R} .
6. Determine the accumulation points of the subset $A = (0, 1)$ of \mathbb{R} .
7. Prove that a subset A of a topological space X is closed if and only if it contains all its accumulation points.
8. Describe a surjective continuous map $f: [0, 1] \rightarrow \Delta$ where $\Delta \subset \mathbb{E}^2$ is a solid equilateral triangle. Explain why f is surjective, stating clearly any theorems that you use in your explanation.
9. Prove that if Y is homeomorphic to a Hausdorff topological space X then Y is Hausdorff.
10. Let \mathbb{Z} be endowed with the cofinite topology. Is \mathbb{Z} Hausdorff with this topology? Exhibit a topology on \mathbb{Z} for which the space is Hausdorff.
11. Prove that a compact subset A of a Hausdorff space X is closed.

10 Simplicial complexes

1. Describe a triangulation on the torus. Determine the number of k -simplices in your triangulation for $k = 0, 1, 2$ and then compute the Euler characteristic.
2. Determine the Euler characteristic of the Möbius band.
3. Determine the Euler characteristic of the sphere $S^n = \{x \in \mathbb{E}^{n+1} : \|x\| = 1\}$ for $n = 1, 2, 3, \dots$

- Describe a triangulation on the double torus.



Determine the number of k -simplices in your triangulation for $k = 0, 1, 2$ and then compute the Euler characteristic.

The second in-class test will consist of a few of the above questions. Questions covered by the first test might also appear on the second test.

11 Homotopy of maps

- Let $Y \subset \mathbb{E}^n$ be an arbitrary convex subset of Euclidean space and let X be an arbitrary topological space. Prove that any two continuous maps $f, g: X \rightarrow Y$ are homotopy equivalent.
- Prove that homotopy equivalence of maps $f \simeq g$ is an equivalence relation on the set of continuous maps $X \rightarrow Y$ from a given space X to a given space Y .
- Let $f: [0, 1] \rightarrow S^1$ be a continuous map. Prove that there is a unique continuous map $\tilde{f}: [0, 1] \rightarrow \mathbb{R}$ such that $f(t) = e(\tilde{f}(t))$ where $e: \mathbb{R} \rightarrow S^1, \theta \mapsto e^{2\pi\theta i}$.
- Define the *winding number* of a map $f: S^1 \rightarrow S^1$ with. Explain why homotopic maps $f \simeq g$ have the same winding number.
- Let $[S^1, S^1]$ denote the set of homotopy classes of maps $S^1 \rightarrow S^1$. Describe a bijection $\omega: [S^1, S^1] \xrightarrow{\cong} \mathbb{Z}$. Explain why ω is onto. Explain why ω is injective.
- State and prove the Fundamental Theorem of Algebra.

12 Homotopy equivalent spaces

- Prove that any convex subspace $Y \subset \mathbb{E}^n$ is homotopy equivalent to the space consisting of a single point.
- Prove that the complex plane minus the origin $\mathbb{C} \setminus \{0\}$ is homotopy equivalent to the circle S^1 .

3. Prove that homotopy equivalence of maps $f \simeq g$ is an equivalence relation on the set of continuous maps $X \rightarrow Y$ from a given space X to a given space Y .
4. Use the fact that the Euler characteristic of a triangulable space is a homotopy invariant to prove Brouwer's fixed point theorem: any continuous map $D^n \rightarrow D^n$ on the closed disc has at least one fixed point.
5. Prove the Frobenius-Perron Theorem: a real square matrix with positive entries has a positive real eigenvalue and the corresponding eigenvector has positive components.

13 Nash Equilibrium

1. Describe what is meant by a *Nash Equilibrium*, explaining any concepts from Game Theory that you use.
2. Use Brouwer's fixed point theorem to prove the existence of a Nash equilibrium in a mixed strategy game.