Given $v_1, \dots, v_n \in \mathbb{R}^p$, $\frac{1}{n}(v_1 + \dots + v_n) = 0$

we set

$$X = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}, \quad C = \frac{1}{n} X X^t$$

The covariance matrix of the transformed data

$$A v_k = \begin{pmatrix} y_{k1} \\ \vdots \\ y_{kp} \end{pmatrix} \qquad 1 \leq k \leq n$$

will be

$$C' = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{pmatrix} \quad \text{with}$$

$$c'_{ii} = \lambda_i = \frac{1}{n} \sum_{k=1}^{n} y_{ki}^2 \qquad (**)$$

$$c'_{ij} = 0 \quad \text{for} \quad i \neq j.$$

If $\lambda_i = 0$ for some i then, from $(**)$,

$y_{ki} = 0$ for all k.

Suppose $\lambda_{q+1} = \lambda_{q+2} = \dots = \lambda_p = 0$ for

some $q \leq p$.

Let B be the $q \times p$ matrix consisting of the first q rows of A. Then the linear map

$$\mathbb{R}^p \to \mathbb{R}^q, \quad v \longmapsto Bv$$

is distance preserving on the set $\{v_1, \dots, v_n\}$.

i.e.

$$d(v_i, v_j) = d(Bv_i, Bv_j)$$

for $1 \leq i, j \leq n$.

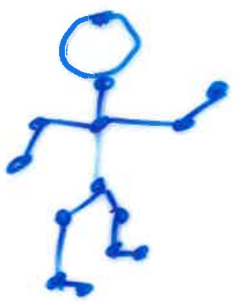
So the geometry of the data $v_1, \dots, v_n \in \mathbb{R}^p$ is the same as the geometry of the data $Bv_1, Bv_2, \dots, Bv_n \in \mathbb{R}^q$.

In practice, we just required $d_{q+1}, \dots, d_p$ to be small, but not necessarily zero. Then

$Bv_1, \dots, Bv_n$ have approximately the same geometry as

$v_1, \dots, v_n$.

<u>Example</u> Gait Analysis



29 sensors placed on a human.

Each sensor measures 3 Euler angles. At a given time t the sensors are represented by a vector $v_t \in \mathbb{R}^{87}$. The human is asked to

i) walk forward along a track

ii) hop " " "

In each case the vectors

$$v_{t_1}, v_{t_2}, \dots, v_{t_n} \in \mathbb{R}^{87}$$

can be viewed using PCA and

$$B : \mathbb{R}^{87} \to \mathbb{R}^3 .$$

We need:

<u>Theorem</u> Let M be a $p \times p$ real symmetric matrix. For column vectors $v \in \mathbb{R}^p$ define

$$f : \mathbb{R}^p \to \mathbb{R}, \ v \mapsto f(v) = v^t M v.$$

Let u be a point on the unit sphere

$$S^{p-1} = \{ v \in \mathbb{R}^p : \|v\| = 1 \}$$

for which $f(u)$ is a maximum for f on the sphere. Then

$$Mu = \lambda u$$

for some $\lambda \in \mathbb{R}$. i.e. u is an eigenvector of M with eigenvalue λ.

Proof

Let $\mathcal{U} = \langle u \rangle$ be the subspace of $\mathbb{R}^p$ spanned by u.

Notation For $w \in \mathbb{R}^p$ write

$$u^t w = u.w$$

Let

$$W = \{ w \in \mathbb{R}^p : u.w = 0 \}$$

be the subspace of vectors orthogonal to u.

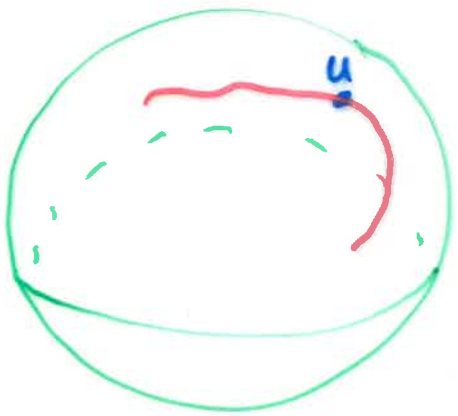
Then $\dim(\mathcal{U}) = 1$

$\dim(W) = p-1$.

For any element $w \in W$ with $\|w\| = 1$ let's define the curve

$$C(t) = (\cos t)\, u + (\sin t)\, w.$$

The curve $c(t)$ lies on the unit sphere



and passes through u. To see this