<u>Theorem</u> Let M be a $p \times p$ real symmetric matrix. For column vectors $v \in \mathbb{R}^p$ define

$$f : \mathbb{R}^p \to \mathbb{R}, \quad v \longmapsto f(v) = v^t M v .$$

Let u be a point on the unit sphere

$$S^{p-1} = \{ v \in \mathbb{R}^p : \|v\| = 1 \}$$

for which $f(u)$ is a maximum for f on the sphere. Then

$$Mu = \lambda u$$

for some $\lambda \in \mathbb{R}$. (i.e. u is an eigenvector of M with eigenvalue λ.)

<u>Proof</u> Let $U = \langle u \rangle$ be the subspace of $\mathbb{R}^p$ spanned by u.

Notation: for $w \in \mathbb{R}^p$ we'll write

$$u . w = u^t w , \quad \|w\| = \sqrt{w^t w}$$

Let

$$W = \{ w \in \mathbb{R}^p : u.w = 0 \}$$

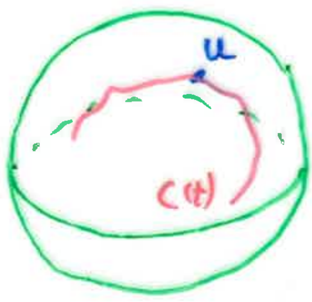
be the <u>subspace</u> of vectors orthogonal to u.

Then $\dim(U) = 1$, $\dim(W) = p-1$.

For any element $w \in W \cap S^{p-1}$ define the curve

$$c(t) = (\cos t) u + (\sin t) w .$$

The curve $c(t)$ lies on the sphere S^{p-1} and passes through u. To see this ...



$$\|c(t)\| = ((\cos t)u^t + (\sin t)w^t)((\cos t)u + (\sin t)w)$$

$$= (\cos^2 t)u^t u + (\sin^2 t)w^t w$$

$$= \cos^2 t + \sin^2 t$$

$$= 1.$$

Also $c(0) = u$.

$$c'(t) = (-\sin t)u + (\cos t)w.$$

Thus, at u (when $t=0$), the direction of the curve is w and the curve is thus perp^dr to u.

Consider

$$g(t) = f(c(t)) = c(t) \cdot Mc(t)$$

Since $f(u)$ is a max, and since $g(0) = f(u)$, it follows that $g'(0) = 0$.

Now

$$g'(t) = c'(t) \cdot Mc(t) + c(t) \cdot Mc'(t)$$

Aside: $w \cdot Mv = w^t Mv$ $\Vert$

$v \cdot Mw = v^t Mw = (v^t Mw)^t$

since M is symm.

Thus $w \cdot Mv = v \cdot Mw$

Any linear homomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ can be represented as

$$\phi(v) = Mv$$

with M some $n \times n$ matrix (w.r.t. standard basis of $\mathbb{R}^n$).

A linear hom^{sm} ϕ is symmetric if

$$(\phi u) \cdot v = u \cdot (\phi v)$$

for all $u, v \in \mathbb{R}^n$. Note that ϕ is symmetric iff the matrix M is symmetric.

The above theorem can be restated as

Theorem Any symmetric linear homomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ has an eigenvector.

Hence

$$g'(t) = 2\, c'(t) \cdot M c(t)$$

$0 = g'(0) = 2\, c'(0) \cdot M c(0) = r w \cdot M u$

with $r \in \mathbb{R}$.

So Mu is perp$^{\text{dr}}$ to w for <u>any</u>

$w \in W$.

This means $Mu = \lambda u$ with $\lambda \in \mathbb{R}$.

$\boxed{}$

Spectral Theorem

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear homom^{sm}. Then $\mathbb{R}^n$ has a basis consisting of eigenvectors of ϕ.

Proof Clearly ϕ has one eigenvector, v say. Let

$V = \langle v \rangle$ be spanned by v.

Let

$$V^\perp = \{ w \in \mathbb{R}^n : w \cdot v = 0 \}$$

ϕ restricts to homomorphisms

i) $\phi : V \longrightarrow V, \quad w \longmapsto \phi(w)$

ii) $\phi : V^\perp \longrightarrow V^\perp, \quad w \longmapsto \phi(w).$

For (i) we note $w = rv, \ r \in \mathbb{R}$

$$\phi(w) = \phi(rv) = r\phi(v) = r\lambda v \in V.$$

For (ii)

$$\begin{aligned} \phi(w) \cdot v &= w \cdot \phi(v) \\ &= w \cdot \lambda v \\ &= \lambda (w \cdot v) = 0. \end{aligned}$$

so $\phi : V^\perp \longrightarrow V^\perp$ can be regarded (!) as a symmetric linear hom^{sm}

$$\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$$

and, as inductive hypothesis, we can assume $V^\perp \cong \mathbb{R}^{n-1}$ has basis consisting of $n-1$ eigenvectors of ϕ. By induction, $\mathbb{R}^n$ has a basis of n eigenvectors of ϕ.

$\boxtimes$