NUI Galway
OÉ Gaillimh

Semester 1 Examinations 2017/2018

Exam Codes	3BA1, 4BA4, 4BCS1, 3OA6, 3BS4, 3BS9
Exams	Third Science and Third Arts Examination
Module	Linear Algebra I
Module Code	MA313, MA3313
External Examiner	Prof Tom Brady
Internal Examiner(s)	Prof Graham Ellis
	Dr Aisling McCluskey
Instructions:	**Answer all four questions**
	All questions carry equal marks.
Duration	2 Hours
No. of Pages	Four pages, including this one
School	Mathematics, Statistics and Applied Mathematics
Requirements	No special requirements
Release to Library:	Yes
Statistical Tables/ Log Tables	No

1. (a) (i) Let U be a vector space with $u_1, u_2, \ldots, u_n \in U$. Explain what is meant by saying that $\{u_1, u_2, \ldots, u_n\}$ is a *spanning set* for U. [3 marks]

(ii) Give an example of a spanning set for the vector space $P_2(t)$, where $P_2(t)$ denotes all polynomials in t with real coefficients that have degree at most 2. [3 marks]

(b) (i) Determine with justification whether the polynomial $p(t) = t$ is a linear combination of polynomials $2t + t^2$ and $2 + t$. [3 marks]

(ii) Hence, or otherwise, determine whether the set $\{2t + t^2, 2 + t\}$ spans $P_2(t)$. [2 marks]

(c) (i) Let V be a vector space, and U a subset of V. Complete the following: *U is a linear subspace of V* if for each u_1, u_2 in U, and each $\cdots$ is an element of U. [2 marks]

(ii) Decide with brief explanation whether $\{(0, y) \in \mathbb{R}^2 : y \geq 0\}$ is a linear subspace of $\mathbb{R}^2$. [3 marks]

(iii) Decide with brief explanation whether $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a - d = 0 \right\}$ is a linear subspace of $M_{2,2}$. [4 marks]

2. (a) (i) Let U be a vector space with $u_1, u_2, \ldots, u_n \in U$. Explain what is meant by saying that $\{u_1, u_2, \ldots, u_n\}$ is *linearly independent*. [2 marks]

(ii) For what values of k is the set $\{(2, 4, 2), (4, 6, 2 + k), (0, 2, 2)\}$ linearly independent in $\mathbb{R}^3$? [3 marks]

(b) (i) Explain what is meant by $\{u_1, u_2, \ldots, u_n\}$ being a *basis* for a vector space U. [2 marks]

(ii) Find a basis for the vector space $\{p(t) \in P_2(t) : p'(4) = p(1)\}$. [3 marks]

(iii) Find the coordinate vector of $t^2 + 7$ with respect to the ordered basis found in part (ii). [2 marks]

(c) Let W be the subspace of $\mathbb{R}^3$ spanned by $\{(1,1,3),(4,4,3),(3,3,0)\}$ and let $A = \begin{pmatrix} 1 & 1 & 3 \\ 4 & 4 & 3 \\ 3 & 3 & 0 \end{pmatrix}$. Find

(i) a basis for W [2 marks]

(ii) the dimension of W [2 marks]

(iii) the rank of A [1 mark]

(iv) a basis for the row space of A [1 mark]

(v) a basis for the column space of A. [2 marks]

3. (a) Three different functions $f : \mathbb{R}^2 \to \mathbb{R}$ are defined below. In each case, determine with explanation whether the given function is linear.

(i) $f(x, y) = (x + y)^2 - (x - y)^2 - 4xy$ [2 marks]

(ii) $f(x, y) = 3x + |y|$ [2 marks]

(iii) $f(x, y) = x$. [2 marks]

(b) (i) Define what is meant by the *kernel* $\ker(g)$ and the *image* $\operatorname{Im}(g)$ of a linear map $g : U \to V$. [3 marks]

(ii) Show that g is injective (or one-one) if and only if $\ker(g) = \{0\}$. [4 marks]

(c) Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ where

$g(x, y, z) = (x + 6y + 5z, -6x + y + 7z, 5x - 6y - 11z)$.

(i) Is $(1, -1, 1)$ in the kernel of g? Is $(6, 1, -6)$ in the image of g? In each case, explain your answer. [3 marks]

(ii) It is known that the dimension of $\mathrm{Im}(g)$ is 3. Use the Rank-Nullity theorem to determine the dimension of $\ker(g)$. Hence, or otherwise, decide with brief explanation whether g is an injective function. [4 marks]

4. (a) Write down the matrix representation of the linear map $D : P_3(t) \to P_2(t)$ given by $D(p(t)) = p'(t)$ with respect to the bases $\{1, t, t^2, t^3\}$ and $\{t - t^2, -2t + t^2, 1 - t\}$ of $P_3(t)$ and $P_2(t)$ respectively. [8 marks]

(b) Find an orthogonal matrix A whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. [5 marks]

(c) Find a matrix representation with respect to the standard basis of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T(v)$ is the orthogonal reflection of v in the plane with equation $x - 2y + 2z = 0$. [7 marks]