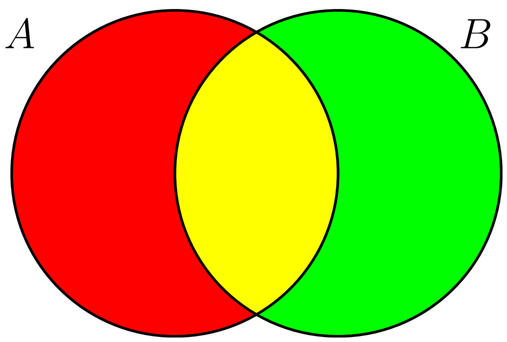
Annotated slides from Wednesday

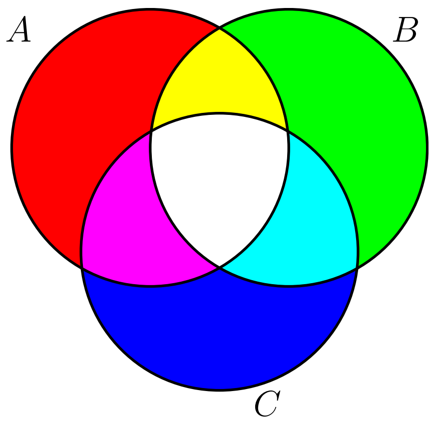
MA204/MA284 : Discrete Mathematics

Week 2: Counting with sets; The Principle of Inclusion and Exclusion (PIE)

Dr Niall Madden

13 & 15 September 2017





Tutorials will start in Week 3 (week beginning 18 September).
You should attend *one of the following sessions*:

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			??		
12 – 1		CA117			
1 – 2					
2 – 3	???		???		
3 – 4	???				
4 – 5					
5 – 6					

If you would like a tutorial at a different time, please indicate your preferences by filling out the form at
https://goo.gl/forms/PMFzVVMAuPXI97Nn1 ← *Link!*

Likely : Tuesday @ 3 (2BCT),

This week...

In this week's classes, we are going to build on the *Additive* and *Multiplicative* Principles from Lecture 2.
After reminding ourselves of the basic ideas, we will present them in the formal setting of *set theory*.

We will then move on to the *Principle of Inclusion/Exclusion*.
The presentation will closely follow Chapter 1 of Levin's *Discrete Mathematics: an open introduction*.

Section 1.1.

Section 1.2

(Friday).

The Additive Principle

If event A can occur m ways, and event B can occur n (disjoint) ways, then event "A or B" can occur in $m + n$ ways.

Example

There are **210** students in registered for Discrete Mathematics, of which **60** are in Engineering, **100** are in Science, and **50** are in Arts.

1. How many ways can be choose a Class Rep who is from **Arts** or **Science**?
2. How many ways can be choose a Class Rep who is from **Arts**, **Science**, or **Engineering**?

1. "Event A" is "Rep is from Arts" and
   "Event B" is "Rep is from Science".

So $m = 50$, $n = 100$. Answer is 150.

These events are disjoint: no one is in both Arts and Science.

The Multiplicative Principle

If event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then event "A **and** B" can occur in $m \times n$ ways.

Example

There are (still) **210** students in registered for Discrete Mathematics, of which **60** are in Engineering, **100** are in Science, and **50** are in Arts.

1. How many ways can be choose two Class Rep, one each from **Arts** and **Science**?
2. How many ways can be choose three Class Rep, one each from **Arts**, **Science**, and **Engineering**?

1. Answer is $(50) \times (100) = 5{,}000$.
2. Answer is $50 \times 100 \times 60 = 300{,}000$.

Again we have applied the M.P. Repeatedly.

Example (Students in Discrete Mathematics (again))

Let D be the set of students in Discrete Mathematics. So $|D| = 210$.
Let E be the set of Discrete Maths students who are in **Engineering**. So $|E| = 60$.
Similarly, let S and A be the sets of Discrete Mathematics students who are in **Science** and **Arts** respectively. So $|S| = 100$, and $|A| = 50$.

What do we mean by... $A \cup E$ is set of everyone in Arts <u>or</u> Engineering. $|A \cup E| = 110$.

(1) ▪ $A \cup E$?

(2) ▪ $A \cap E$?

$A \cap E$ is everyone in <u>both</u> Arts & Engineering. $|A \cap E| = 0$.

ie $A \cap E = \emptyset$ (The empty set)

"A and E are disjoint".

1 **Additive Principle in terms of "events"**

If event A can occur m ways, and event B can occur n (disjoint/independent) ways, then event "A **or** B" can occur in $m + n$ ways.

But an "event" can be expressed as just selecting an element of a set. For example, the event "*Choose a Class Rep from Arts*" is the same as "*Choose an element of the set A*". Also, saying

- **event A can occur m ways**, as the same as saying $|A| = m$,
- **event B can occur n ways**, as the same as saying $|B| = n$,
- events A and B are disjoint/independent means $|A \cap B| = 0$ (or, equivalently $A \cap B = \emptyset$).

2. **Additive Principle for Sets**

Given two sets A and B with $|A| = m$, $|B| = n$ and $|A \cap B| = 0$. Then

$$|A \cup B| = m + n.$$

Additive Principle for Sets

Given two sets A and B with $|A \cap B| = 0$. Then

$$|A \cup B| = |A| + |B|.$$

Example:

$A = \{a, b, c\}$ $B = \{x, y\}$

$A \cup B = \{a, b, c, x, y\}$ $A \cap B = \emptyset$.

and $|A| = 3$, $|B| = 2$, $|A \cup B| = 5$.

The **Cartesian Product** of sets A and B is

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

This is the set of pairs where the first term in each pair comes from A, **and** the second comes from B.

Example

Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4\}$.
Write down $A \times B$ and $A \times C$.

$$A \times B = \{(1,1), (1,3), (1,5), (2,1), (2,3), (2,5), (3,1), (3,3), (3,5)\}$$

$$|A \times B| = 9 = 3 \times 3 = |A| \cdot |B|.$$

$$A \times C = \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)\}.$$

$$|A \times C| = 6 = |A| \cdot |C|$$

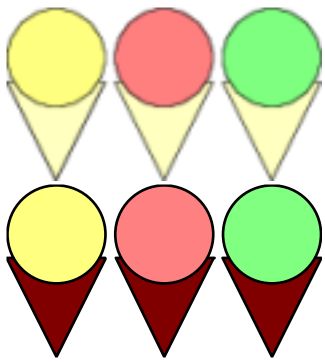
If $|A| = m$ and $|B| = n$, then $|A \times B| = m \cdot n$.
Why?

When we write out all the elements as a grid, there will be
m rows and
n columns.
So a total of $m \cdot n$ entries.

What has the *Cartesian Product* got to do with the **Multiplicative Principle**?

Consider the following example... Suppose we go to our favourite ice-cream shop where they stock three flavours: **V**anilla, **Strawberry** and **M**int.
They had two types of cone: plain **C**ones and **W**affle cones.
How many ways can I place an order (for 1 cone and 1 scoop?).

$A = \{V, S, M\}$
(set of flavours)



$B = \{C, W\}$.
(Types of cone)

$A \times B = \{(V,C), (S,C), (M,C), (V,W), (S,W), (M,W)\}$

$|A \times B| = 6.$

Last week we learned:

The Multiplicative Principle

If event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then event "A **and** B" can occur in $m \times n$ ways.

We can now express this in terms of sets:

Multiplicative Principle for Sets

Given two sets A and B,

$$|A \times B| = |A| \cdot |B|.$$

This extends to three or more sets in the obvious way:

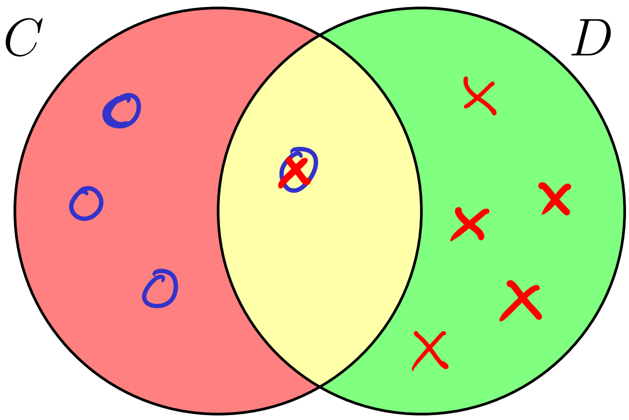
$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

$$|A \times B \times C| = |A \times B| \cdot |C| = |A| \cdot |B| \cdot |C|$$

Good news!

Remember from last week that the NUIG Animal Shelter had 4 cats and 6 dogs in need of a home. Well, they have all been adopted and, (unsurprisingly, given their kind and generous nature) by Discrete Mathematics students. They went to 9 different homes, because one person adopted both a cat and a dog.

C = set of "cat homes"



D = set of dog homes

$$|C \cup D| = |C| + |D| - |C \cap D|$$

$$9 = 4 + 6 - 1$$

Since we admire those people that adopted an animal so much, we want one of them as our Class Rep. That is we will choose our Class Rep from one of the sets C and D where $|C| = 4$ and $|D| = 6$.

If we were to apply the **Additive Principle** *naïvly*, we would think that we have $|C| + |D| = 10$ choices for our Rep. But of course, we only have $|C \cup D| = 9$ choices.

This is often expressed as follows:

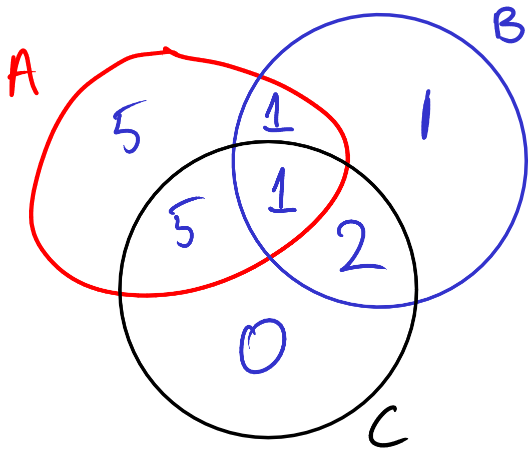
$$|C \cup D| = |C| + |D| - |C \cap D|.$$

Example (See Example 1.1.8 of textbook)

An examination in three subjects, **A**lgebra, **B**iology, and **C**hemistry, was taken by 41 students. The following table shows how many students failed in each single subject and in their various combinations.

Subject:	A	B	C	A&B	A&C	B&C	A&B&C
Failed:	12	5	8	2	6	3	1

How many students failed at least one subject?



$|A| = 12$ $|B| = 5$ $|C| = 8$

$|A \cap B| = 2$

Ans: 15

[Finished here 13/09/2017]

This example shows how to extend the Principle in Inclusion/Exclusion to three sets, A, B and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$