MA204/MA284 : Discrete Mathematics

Week 2: Counting with sets; The Principle of Inclusion and Exclusion (PIE)

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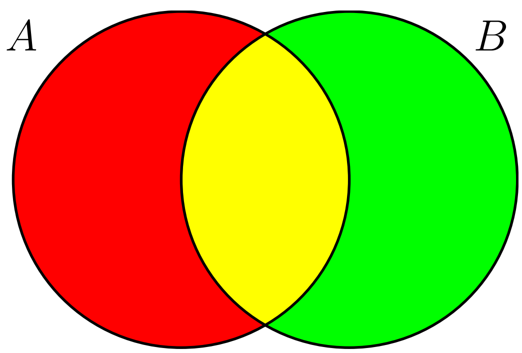
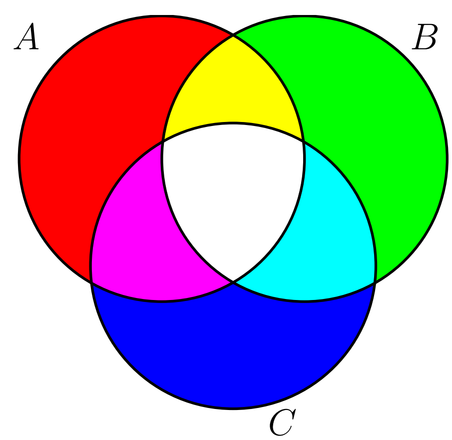


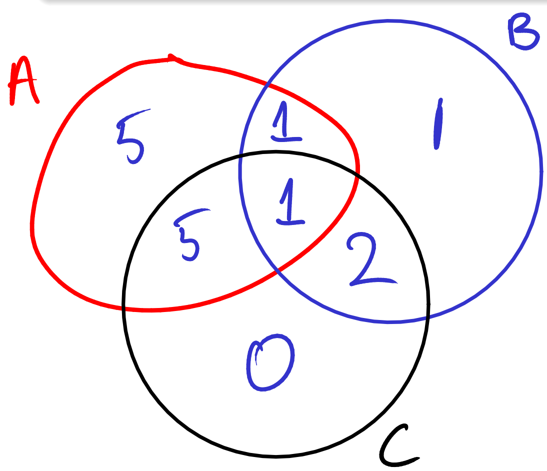


Example (See Example 1.1.8 of textbook)

An examination in three subjects, **A**lgebra, **B**iology, and **C**hemistry, was taken by 41 students. The following table shows how many students failed in each single subject and in their various combinations.

Subject:	A	B	C	A&B	A&C	B&C	A&B&C
Failed:	12	5	8	2	6	3	1

How many students failed at least one subject?



$|A| = 12$ $|B| = 5$ $|C| = 8$

$|A \cap B| = 2$

Ans: 15

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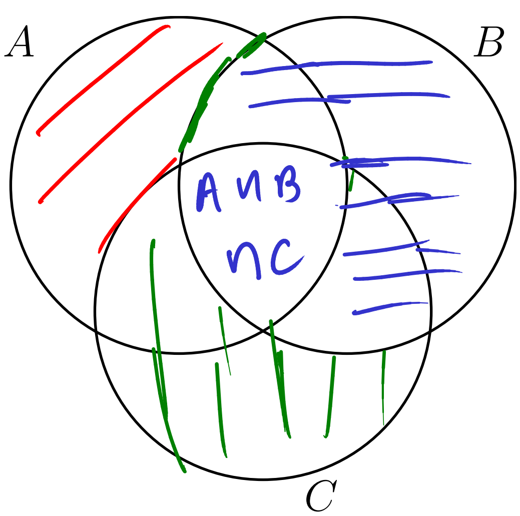
The Principle of Inclusion and Exclusion (PIE)

This example shows how to extend the Principle in Inclusion/Exclusion to three sets, A, B and C:

$$\underbrace{|A \cup B \cup C|}_{\text{People who failed at least one paper}} = \underset{12}{|A|} + \underset{5}{|B|} + \underset{8}{|C|} - \underset{2}{|A \cap B|} - \underset{6}{|A \cap C|} - \underset{3}{|B \cap C|} + |A \cap B \cap C|.$$



The question we will investigate is:

How many subsets are there are $A_1 = \{1\}$?
How many subsets are there are $A_2 = \{1, 2\}$?
How many subsets are there are $A_3 = \{1, 2, 3\}$?
How many subsets are there are $A_4 = \{1, 2, 3, 4\}$?

$$\vdots$$

How many subsets are there are $A_k = \{1, 2, 3, \dots, k\}$?

Here is another way of expressing this:

Power set

Let $P(A)$ be the POWER SET of A, i.e., the set of all subsets of A, including the empty set.
What is $|P(A)|$?

We'll answer this question in two different ways, which is a classic approach to problems in combinatorics.

First we'll list all the subsets of A_1, A_2 and A_3, and try to guess the answer. Then we will try to explain it.

The subsets of $A_1 = \{1\}$, ie $P(A_1)$ are
$\emptyset$, $\{1\}$. So $|P(A_1)| = 2 = 2^1$

The subsets of $A_2 = \{1,2\}$ are
$\emptyset$,
$\{1\}$, $\{2\}$ so $P(A_2) = 4 = 2^2$
$\{1,2\}$

Also, (see OHP), $|P(A_3)| = 8 = 2^3$

Looks like $P(A_4) = 16 = 2^4$, $P(A_k) = 2^k$.

Here is another approach. Consider $P(A_2) = P(\{1, 2\})$.

When constructing a subset, we can proceed as follows:

- **Event A:** choose to include the element **1** or not. This can happen in 2 ways.
- **Event B:** choose to include the element **2** or not. This can happen in 2 ways.

Now apply the multiplicative principle.

$A_2 = \{1, 2\}$. Every subset either contains "1" OR not. (Event A).

and every subset contains 2 or not: Event B.

Example

How many subsets are there are $A_5 = \{1, 2, 3, 4, 5\}$?

we could list them all, but would take too long.

Instead: apply multiplicative principle, to show there are

$$2 \times 2 \times 2 \times 2 \times 2 = \underline{\underline{32}}$$

subsets.

Here is a slightly harder problem

How many subsets are there are $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

We will look at **three** different ways of answering this question:

1. By "brute-force": simply listing all the possibilities.
2. By counting all sets that **do not** have three elements.
3. Next week, by using **binomial coefficients**.

List Them all:

$\{1,2,3\}$, $\{1,3,4\}$ $\{2,3,4\}$

$\{1,2,4\}$, $\{1,3,5\}$ $\{2,3,5\}$ $\{3,4,5\}$

$\{1,2,5\}$, $\{1,4,5\}$ $\{2,4,5\}$

Answer: 10

Method 2

How many subsets are there are $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

Here is an easy way of answering this question.

- How many subsets of A_5 have no elements? 1 (Empty set).
- How many subsets of A_5 have 5 elements? 1 $\{1,2,3,4,5\}$.
- How many subsets of A_5 have 1 element? 5.
- How many subsets of A_5 have 4 elements? 5.
- Now use that the number of subsets of A_5 with **3** elements, is the same as the number with **2** elements. Eg $\{1,3,5\}$ corresponds to $\{2,4\}$.

There are 32 subsets of A_5 in total.

1 has no elements, 1 has 5 elements
5 has 1 " , 5 has 4 " .

So the number with 2 or 3 is $32 - (1+1+5+5)$
$= 20$.

Then, since there are the same number of subsets with 2 as with 3 elements, answer is $20/2 = 10$.

Exercises

Here are a set of exercises to help you work through the material presented during Week 2.

Except where indicated, all these exercises are taken from Section 1.1 of textbook (Levin's Discrete Mathematics - an open introduction).

Q1. We usually write numbers in decimal form (i.e., base 10), meaning numbers are composed using 10 different "digits" $\{0, 1, \ldots, 9\}$. Sometimes though it is useful to write numbers *hexadecimal* or base 16. Now there are 16 distinct digits that can be used to form numbers: $\{0, 1, \ldots, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$. So for example, a 3 digit hexadecimal number might be 2B8.

(a) How many 2-digit hexadecimals are there in which the first digit is E or F? Explain your answer in terms of the additive principle (using either events or sets).

(b) Explain why your answer to the previous part is correct in terms of the multiplicative principle (using either events or sets). Why do both the additive and multiplicative principles give you the same answer?

(c) How many 3-digit hexadecimals start with a letter (A-F) and end with a numeral (0-9)? Explain.

(d) How many 3-digit hexadecimals start with a letter (A-F) or end with a numeral (0-9) (or both)? Explain.

Exercises

Q2. A group of students were asked about their TV watching habits. Of those surveyed,

- 28 students watch *The Walking Dead*,
- 19 watch *The Blacklist*, and
- 24 watch *Game of Thrones*.
- Additionally, 16 watch *The Walking Dead* and *The Blacklist*,
- 14 watch *The Walking Dead* and *Game of Thrones*,
- and 10 watch *The Blacklist* and *Game of Thrones*.
- There are 8 students who watch all three shows.

How many students surveyed watched at least one of the shows?

Q3. In a recent survey, 30 students reported whether they liked their potatoes Mashed, French-fried, or Baked. 15 liked them mashed, 20 liked French fries, and 9 liked twice baked potatoes. Additionally, 12 students liked both mashed and fried potatoes, 5 liked French fries and baked potatoes, 6 liked mashed and baked, and 3 liked all three styles. How many students do *not* like potatoes? Explain why your answer is correct.

Q4. For how many $n \in \{1, 2, \ldots, 500\}$ is n a multiple of one or more of 5, 6, or 7?

Finished here Friday.