MA204/MA284 : Discrete Mathematics

Week 2: Counting with sets; The Principle of Inclusion and Exclusion (PIE)

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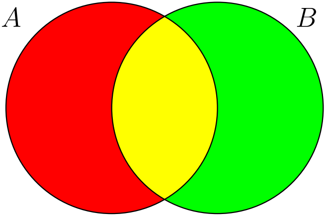
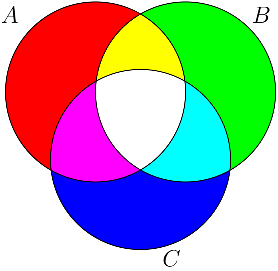
13 & 15 September 2017





Tutorials will start in Week 3 (week beginning 18 September).
You should attend *one of the following sessions*:

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			??		
12 – 1		CA117			
1 – 2					
2 – 3	???		???		
3 – 4	???				
4 – 5					
5 – 6					

If you would like a tutorial at a different time, please indicate your preferences by filling out the form at
`https://goo.gl/forms/PMFzVVMAuPXI97Nn1` ← *Link!*

In this week's classes, we are going to build on the *Additive* and *Multiplicative* Principles from Lecture 2.

After reminding ourselves of the basic ideas, we will present them in the formal setting of *set theory*.

We will then move on to the *Principle of Inclusion/Exclusion*.

The presentation will closely follow Chapter 1 of Levin's *Discrete Mathematics: an open introduction*.

The Additive Principle

If event A can occur m ways, and event B can occur n (disjoint) ways, then event "A **or** B" can occur in $m + n$ ways.

Example

There are **210** students in registered for Discrete Mathematics, of which **60** are in Engineering, **100** are in Science, and **50** are in Arts.

1. How many ways can be choose a Class Rep who is from **Arts** or **Science**?
2. How many ways can be choose a Class Rep who is from **Arts**, **Science**, or **Engineering**?

The Multiplicative Principle

If event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then event "A **and** B" can occur in $m \times n$ ways.

Example

There are (still) **210** students in registered for Discrete Mathematics, of which **60** are in Engineering, **100** are in Science, and **50** are in Arts.

1. How many ways can be choose two Class Rep, one each from **Arts** and **Science**?
2. How many ways can be choose three Class Rep, one each from **Arts**, **Science**, and **Engineering**?

Example (Students in Discrete Mathematics (again))

Let D be the set of students in Discrete Mathematics. So $|D| = 210$.
Let E be the set of Discrete Maths students who are in **Engineering**. So $|E| = 60$.
Similarly, let S and A be the sets of Discrete Mathematics students who are in **Science** and **Arts** respectively. So $|S| = 100$, and $|A| = 50$.

What do we mean by...

- $A \cup E$?
- $A \cap E$?

Additive Principle in terms of "events"

If event A can occur m ways, and event B can occur n (disjoint/independent) ways, then event "A **or** B" can occur in $\boldsymbol{m+n}$ ways.

But an "event" can be expressed as just selecting an element of a set. For example, the event "*Choose a Class Rep from Arts*" is the same as "*Choose an element of the set A*". Also, saying

- **event** A **can occur** m **ways**, as the same as saying $|A| = m$,
- **event** B **can occur** n **ways**, as the same as saying $|B| = n$,
- events A and B are disjoint/independent means $|A \cap B| = 0$ (or, equivalently $A \cap B = \emptyset$).

Additive Principle for Sets

Given two sets A and B with $|A| = m$, $|B| = n$ and $|A \cap B| = 0$. Then

$$|A \cup B| = m + n.$$

Additive Principle for Sets

Given two sets A and B with $|A \cap B| = 0$. Then

$$|A \cup B| = |A| + |B|.$$

Example:

The **Cartesian Product** of sets A and B is

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

This is the set of pairs where the first term in each pair comes from A, **and** the second comes from B.

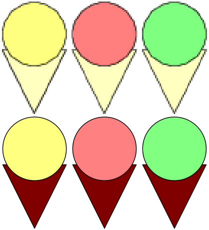
Example

Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4\}$.
Write down $A \times B$ and $A \times C$.

If $|A| = m$ and $|B| = n$, then $|A \times B| = m \cdot n$.
Why?

What has the *Cartesian Product* got to do with the **Multiplicative Principle**?

Consider the following example... Suppose we go to our favourite ice-cream shop where they stock three flavours: **V**anilla, **Strawberry** and **M**int.
They had two types of cone: plain **C**ones and **W**affle cones.
How many ways can I place an order (for 1 cone and 1 scoop?).



Last week we learned:

The Multiplicative Principle

If event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then event "A **and** B" can occur in $\boldsymbol{m} \times \boldsymbol{n}$ ways.

We can now express this in terms of sets:

Multiplicative Principle for Sets

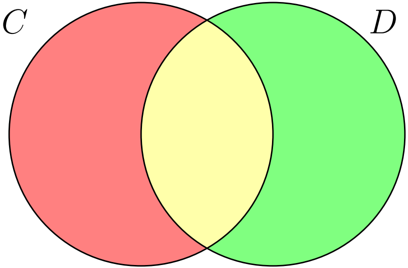
Given two sets A and B,

$$|A \times B| = |A| \cdot |B|.$$

This extends to three or more sets in the obvious way:

Good news!

Remember from last week that the NUIG Animal Shelter had 4 cats and 6 dogs in need of a home. Well, they have all been adopted and, (unsurprisingly, given their kind and generous nature) by Discrete Mathematics students. They went to 9 different homes, because one person adopted both a cat and a dog.



Since we admire those people that adopted an animal so much, we want one of them as our Class Rep. That is we will choose our Class Rep from one of the sets C and D where $|C| = 4$ and $|D| = 6$.

If we were to apply the **Additive Principle** *naïvly*, we would think that we have $|C| + |D| = 10$ choices for our Rep. But of course, we only have $|C \cup D| = 9$ choices.

This is often expressed as follows:

$$|C \cup D| = |C| + |D| - |C \cap D|.$$

Example (See Example 1.1.8 of textbook)

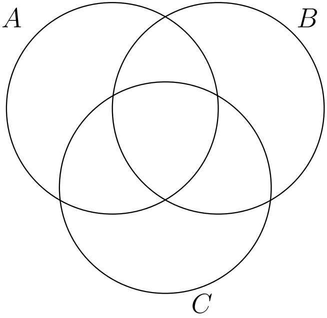
An examination in three subjects, **A**lgebra, **B**iology, and **C**hemistry, was taken by 41 students. The following table shows how many students failed in each single subject and in their various combinations.

Subject:	A	B	C	A&B	A&C	B&C	A&B&C
Failed:	12	5	8	2	6	3	1

How many students failed at least one subject?

This example shows how to extend the Principle in Inclusion/Exclusion to three sets, A, B and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$



The question we will investigate is:

How many subsets are there are $A_1 = \{1\}$?
How many subsets are there are $A_2 = \{1, 2\}$?
How many subsets are there are $A_3 = \{1, 2, 3\}$?
How many subsets are there are $A_4 = \{1, 2, 3, 4\}$?
$\vdots$
How many subsets are there are $A_k = \{1, 2, 3, \dots, k\}$?

Here is another way of expressing this:

Power set

Let $P(A)$ be the POWER SET of A, i.e., the set of all subsets of A, including the empty set.
What is $|P(A)|$?

We'll answer this question in two different ways, which is a classic approach to problems in combinatorics.

First we'll list all the subsets of A_1, A_2 and A_3, and try to guess the answer. Then we will try to explain it.

Here is another approach. Consider $P(A_2) = P(\{1, 2\})$.

When constructing a subset, we can proceed as follows:

- **Event A:** choose to include the element **1** or not. This can happen in 2 ways.
- **Event B:** choose to include the element **2** or not. This can happen in 2 ways.

Now apply the multiplicative principle.

Example

How many subsets are there are $A_5 = \{1, 2, 3, 4, 5\}$?

Here is a slightly harder problem

How many subsets are there are $A = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

We will look at **three** different ways of answering this question:

1. By "brute-force": simply listing all the possibilities.
2. By counting all sets that **do not** have three elements.
3. Next week, by using **binomial coefficients**.

Method 2

How many subsets are there are $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

Here is an easy way of answering this question.

- How many subsets of A_5 have no elements?
- How many subsets of A_5 have 5 elements?
- How many subsets of A_5 have 1 element?
- How many subsets of A_5 have 4 elements?

Here are a set of exercises to help you work through the material presented during Week 2.

Except where indicated, all these exercises are taken from Section 1.1 of textbook (Levin's Discrete Mathematics - an open introduction).

Q1. We usually write numbers in decimal form (i.e., base 10), meaning numbers are composed using 10 different "digits" $\{0, 1, \dots, 9\}$. Sometimes though it is useful to write numbers *hexadecimal* or base 16. Now there are 16 distinct digits that can be used to form numbers: $\{0, 1, \dots, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$. So for example, a 3 digit hexadecimal number might be 2B8.

(a) How many 2-digit hexadecimals are there in which the first digit is E or F? Explain your answer in terms of the additive principle (using either events or sets).

(b) Explain why your answer to the previous part is correct in terms of the multiplicative principle (using either events or sets). Why do both the additive and multiplicative principles give you the same answer?

(c) How many 3-digit hexadecimals start with a letter (A-F) and end with a numeral (0-9)? Explain.

(d) How many 3-digit hexadecimals start with a letter (A-F) or end with a numeral (0-9) (or both)? Explain.

Q2. A group of students were asked about their TV watching habits. Of those surveyed,

- 28 students watch *The Walking Dead*,
- 19 watch *The Blacklist*, and
- 24 watch *Game of Thrones*.
- Additionally, 16 watch *The Walking Dead* and *The Blacklist*,
- 14 watch *The Walking Dead* and *Game of Thrones*,
- and 10 watch *The Blacklist* and *Game of Thrones*.
- There are 8 students who watch all three shows.

How many students surveyed watched at least one of the shows?

Q3. In a recent survey, 30 students reported whether they liked their potatoes Mashed, French-fried, or Baked. 15 liked them mashed, 20 liked French fries, and 9 liked twice baked potatoes. Additionally, 12 students liked both mashed and fried potatoes, 5 liked French fries and baked potatoes, 6 liked mashed and baked, and 3 liked all three styles. How many students do *not* like potatoes? Explain why your answer is correct.

Q4. For how many $n \in \{1, 2, \ldots, 500\}$ is n a multiple of one or more of 5, 6, or 7?

Q5. (MA204 Semester 1 Exam, 2014/2015) In the set of positive integers $\{1, 2, 3, ..., 240\}$, how many of the numbers are **not** divisible by any of the primes 2, 3, 5?

How many of the numbers in the set are divisible by precisely two of the primes 2, 3, 5?

Q6. Let A, B, and C be sets.

(a) Find $|(A \cup C) \setminus B|$ provided $|A| = 50$, $|B| = 45$, $|C| = 40$, $|A \cap B| = 20$, $|A \cap C| = 15$, $|B \cap C| = 23$, and $|A \cap B \cap C| = 12$.

(b) Describe a set in terms of A, B, and C with cardinality 26.