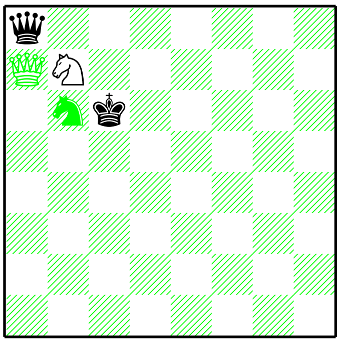
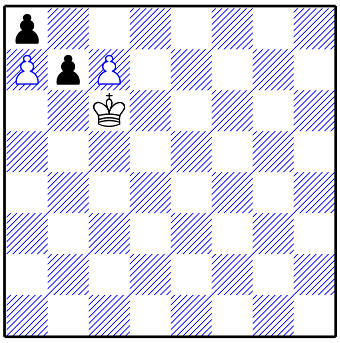




MA204/MA284 : Discrete Mathematics

Week 3: Binomials Coefficients

Dr Niall Madden

20 & 22 September 2017

These slides are based on §1.2 of Oscar Levin’s *Discrete Mathematics: an open introduction*.

Tutorials (other than Tuesdays at 3) started this week. You should attend one of the sessions listed below.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			AdB-1020		
12 – 1		CA117			
1 – 2					
2 – 3	Tyndall		AC213		
3 – 4	IT202	*TBA*			
4 – 5					
5 – 6					

ASSIGNMENT 1 is now open!

1718-MA284

To access the assignment, go to
~~https://webwork.nuigalway.ie/webwork2/1617-MA284~~
http://mathswork.nuigalway.ie/

Your USERNAME is:
id number

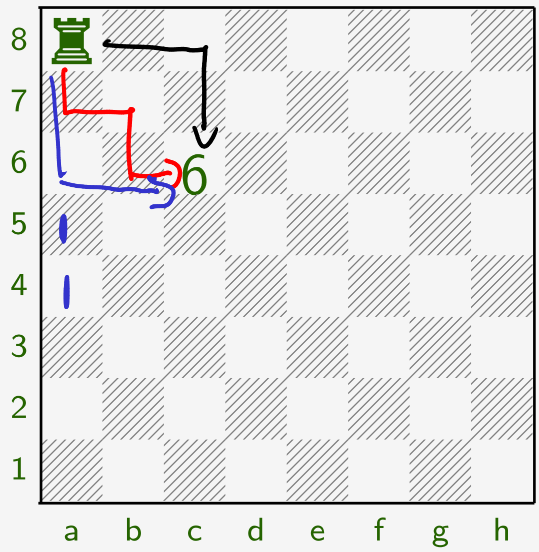
Your PASSWORD is: id number.

There are 20 questions.
You may attempt each one up to ~~10~~ 20 times.
This assignment contributes 10% to your final grade for Discrete Mathematics.

Deadline: 5pm, Thursday 6th October.

Help: Tutorials or SUMS.

A rook can move only in straight lines (not diagonally). *Fill in each square of the chess board below with the number of different shortest paths the rook in the upper left corner can take to get to the square, moving one space at a time.* For example, there are **six** paths from the rook to the square **c6**: DDRR, DRDR, DRRD, RDDR, RDRD, and RRDD. (R = *right*, D = *down*).



bit string of lenght 4.

A **bit** is a "binary digits" (i.e., 0 or 1).
A **bit string** is a string (list) of bits, e.g. 1001, 0, 111111, 10101010.

The *length* of the string is the number of bits.
A n-bit string has length n.
The set of all n-bit strings (for given n) is denoted $\mathbf{B}^n$.

Each bit string is a binary number.

Examples:

$B^1 = \{0, 1\}$ $\qquad$ $B^2 = \{00, 01, 10, 11\}$

$$B^3 = \left\{ \begin{array}{cccc} \overset{0}{000}, & \overset{1}{001}, & \overset{2}{010}, & \overset{3}{011}, \\ \underset{4}{100}, & \underset{5}{101}, & \underset{6}{110}, & \underset{7}{111} \end{array} \right\}$$

$|B^1| = 2$ $\qquad$ $|B^2| = 4$ $\qquad$ $|B^3| = 8$

The *weight* of the string is the number of 1's. (Also: sum of the digits)
The set of all n-bit strings of weight k is denoted $\mathbf{B}^n_k$.

Examples:

$B^1_0 = \{0\}$ $B^1_1 = \{1\}$.

$B^2_0 = \{00\}$ $B^2_1 = \{01, 10\}$ $B^2_2 = \{11\}$

$B^3_0 = \{000\}$ $|B^3_0| = 1$

$B^3_1 = \{001, 010, 100\}$ $|B^3_1| = 3$

$B^3_2 = \{011, 101, 110\}$ $|B^3_2| = 3$

$B^3_3 = \{111\}$. $|B^3_3| = 1$

$|B^3| = 8$

Bit strings

- The set of all n-bit strings (for given n) is denoted $\mathbf{B}^n$.
- The set of all n-bit strings of weight k is denoted $\mathbf{B}^n_k$.

Some counting questions:

1. How many bit strings are there of length 5? That is, what is $|\mathbf{B}^5|$?
2. Of these, how many have weight 3? That is, what is $|\mathbf{B}^5_3|$?

1. Ans is 32. There are 5 bits in each string, and for each one we have 2 choices: 0 or 1.
By the multiplicative principle

$$|B^5| = 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$$

Bit strings

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Some counting questions:

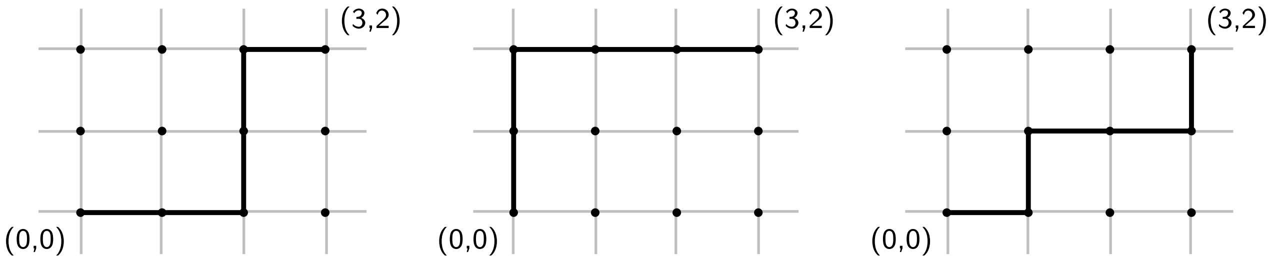
1. How many bit strings are there of length 5? That is, what is $|\mathbf{B}^5|$?
2. Of these, how many have weight 3? That is, what is $|\mathbf{B}^5_3|$?

2. This is a little harder. However, observe that every bit string begins with 0 or 1. The number of string in B^5_3 that start with 0 is $|B^4_3|$. And the number that start with 1 is $|B^4_2|$. So $|B^5_3| = |B^4_3| + |B^4_2|$.
$= |B^3_3| + |B^3_2| + |B^3_2| + |B^3_1| = 10$.

The (integer) *lattice* is the set of all points in the Cartesian plane for which both the x and y coordinates are integers.

A *lattice path* is a **shortest possible path** connecting two points on the lattice, moving only horizontally and vertically.

Example: three possible lattice paths from the points $(0,0)$ to $(3,2)$ are:



RRUUR UURRR RURRU

Question: How many lattice paths are there from $(0,0)$ to $(3,2)$?

Useful observation 1

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as $|B^5_3|$.

Why? Each path can be expressed as a string of five R's and U's, with 3 R's and 2 U's. Same as bit string in B^5_3.

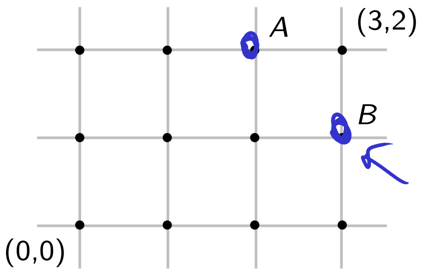
Useful observation 2

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as the number from $\underbrace{(0,0) \text{ to } (2,2)}_{A}$, plus the number from $(0,0)$ to $(3,1)$.

B

So, again

$$|B^5_3| = |B^4_2| + |B^4_3|$$



There are $|B^4_3|$

Version 1

What is the coefficient of (say) x^3y^2 in $(x+y)^5$?

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \checkmark$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \checkmark$$
$$(x+y)^5 = x^5 + 5x^4y + \boxed{10x^3y^2} + 10x^2y^3 + 5xy^4 + y^5$$

So, by doing a lot of multiplication, we have worked out that the coefficient of x^3y^2 is 10 (which is rather familiar....)

But, not surprisingly there this a more systematic way of answering this problem.

"Easier"

Version 2

What is the coefficient of (say) x^3y^2 in $(x+y)^5$?

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y).$$

We can work out the coefficient of x^3y^2 in the expansion of $(x+y)^5$ by counting the number of ways we can choose three x's and two y's in

$$(x+y)(x+y)(x+y)(x+y)(x+y).$$

$x\,x\,x\,y\,y$ $\quad$ $x\,y\,x\,y\,x$

$= x^3y^2$ $\quad$ $1\;0\;1\;0\;1$

$1\;1\;1\;0\;0$

Each term contributes an x or a y:

Same as $|B^5_3|$.

These numbers that occurred in all our examples are called *binomial coefficients*, and are denoted $\binom{n}{k}$

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as ``}n \text{ choose } k\text{''}$$

1. $\binom{n}{k} = |\mathbf{B}^n_k|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n, each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
5. $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects.

Finished here wednesday.