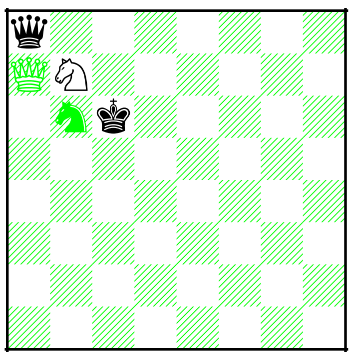
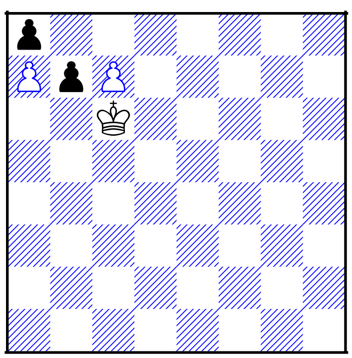
Annotated notes from Friday





MA204/MA284 : Discrete Mathematics

Week 3: Binomials Coefficients

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These slides are based on §1.2 of Oscar Levin's *Discrete Mathematics: an open introduction*.

Calculating binomial coefficients

If we were to skip ahead we would learn that there is a formula for

$$\binom{n}{k} \qquad \text{(that is, ``n choose k'')}$$

that is expressed in terms of **factorials**.

Recall that the *factorial* of a natural number, n is

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1.$$

Examples: $1! = 1$ $\quad 2! = 2 \times 1 = 2$ $\quad 3! = 3 \times 2 \times 1 = 6$ $\quad 4! = 4 \times 3 \times 2 \times 1 = 24$

$10! = 3{,}628{,}800$ $\quad 14! \doteq 87.2$ billion (to compare age of the universe in years is 13 billion).

$18! \doteq 6.4 \times 10^{15} \cong$ distance in meters, light travels in one year.

$59! \cong \begin{cases} \text{number of ways of ranking the 2BCT students.} \\ \text{number of particles in observable universe.} \end{cases}$

We will eventually learn that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples $\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{(3 \cdot 2 \cdot 1)}(2 \cdot 1)} = 10.$

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(2)(2)} = 6$$

$$\binom{4}{3} = \frac{4!}{3!\,1!} = 4$$

Note $4 + 6 = 10.$

Calculating binomial coefficients

However, the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is not very useful in practice.

Example

There are exactly(!) 200 students in this Discrete Mathematics class. Of those, ~~17~~ 25 are Arts students. How many other subsets of size ~~17~~ 25 are there?

Answer: 1.8308×10^{24}. But this is not easy to compute...

$$\binom{200}{25} = \frac{200!}{(25)!\,(175)!} = \frac{7.886 \times 10^{374}}{\dots}$$

This is very hard to compute, using $\frac{n!}{k!(n-k)!}$ given the size of the numbers.

Earlier, we learned that if the set of all n-bit strings with weight k is written $\mathbf{B}^n_k$, then

$$|\mathbf{B}^n_k| = |\mathbf{B}^{n-1}_{k-1}| + |\mathbf{B}^{n-1}_k|.$$

Similarly, we get find that...

Recurrence relation for $\binom{n}{k}$

Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Why: Suppose we wished to pick a group of k people from this class of n people. This can be done in $\binom{n}{k}$ ways. If that group must contain Filipe, there are $\binom{n-1}{k-1}$. If that group does not contain Filipe, there are $\binom{n-1}{k}$. By the additive principle, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Recurrence relation for $\binom{n}{k}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This is often presented as *Pascal's Triangle*

$\binom{0}{0} = 1$

$\binom{1}{0} = 1$ $\quad\quad$ $\binom{1}{1} = 1$

$\binom{2}{0} = 1$ $\quad\quad$ $\binom{2}{1} = 2$ $\quad\quad$ $\binom{2}{2} = 1$

$\binom{3}{0} = 1$ $\quad\quad$ $\binom{3}{1} = 3$ $\quad\quad$ $\binom{3}{2} = 3$ $\quad\quad$ $\binom{3}{3} = 1$

$\binom{4}{0}$ 1 $\quad\quad$ $\binom{4}{1} = 4$ $\quad\quad$ $\binom{4}{2} = 6$ $\quad\quad$ $\binom{4}{3} = 4$ $\quad\quad$ $\binom{4}{4}$ ~ 1,

Source: http://www.flickr.com/photos/35652310@N00/4139577452/.

Example

The NUIG Animal Shelter has 4 cats.

(a) How many choices do we have for a single cat to adopt? $\binom{4}{1} = 4$

(b) How many choices do we have if we want to adopt two cats? $\binom{4}{2} = 6$

(c) How many choices do we have if we want to adopt three cats? $\binom{4}{3} = 4$

(d) How many choices do we have if we want to adopt four cats? $\binom{4}{4} = 1$

Note: choosing 1 to keep is the same as choosing 3 not to keep $\binom{4}{1} = \binom{4}{3}$.

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

Example: List all permutations of the letters A, R and T?

ART, ATR,
RAT RTA
TAR TRA

$\Bigg\}$ 6 in total.

Important: order matters - "ART" $\neq$ "TAR" $\neq$ "RAT".

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

We can also count the number of permutations of the letters `A`, `R` and `T`, without listing them:

Use multiplication principle:

There are 3 choices for first letter,

and 2 choices for 2^{nd} letter

and $\underline{1}$ for 3^{rd} letter

$$3! = 3 \cdot 2 \cdot 1 = 6$$

More generally, recall that $n!$ (read "n factorial") is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

E.g.,

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720.$$

$$10! = 3,628,800, \quad 20! = 2,432,902,008,176,640,000 \approx 2.43 \times 10^{18}.$$

no two are same.

There are $n!$ (i.e., n factorial) permutations of n (distinct) objects.

Why: There are n choices for 1^{st} position
& $(n-1)$ " " 2^{nd} "
& $(n-2)$ " " 3^{rd} "
⋮
1 choice for n^{th} position

Total: $n(n-1)(n-2)\cdots(2)(1) = n!$

Exercises

Q1. Let $S = \{1, 2, 3, 4, 5, 6\}$

(a) How many subsets are there total?
(b) How many subsets have $\{2, 3, 5\}$ as a subset?
(c) How many subsets contain at least one odd number?
(d) How many subsets contain exactly one even number?
(e) How many subsets are there of cardinality 4?
(f) How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?
(g) How many subsets of cardinality 4 contain at least one odd number?
(h) How many subsets of cardinality 4 contain exactly one even number?

Q2. How many subsets of $\{0, 1, \ldots, 9\}$ have cardinality 6 or more? (Hint: Break the question into five cases).

Q3. How many shortest lattice paths start at (3,3) and end at (10,10)?
How many shortest lattice paths start at (3,3), end at (10,10), and pass through (5,7)?

Q4. Suppose you are ordering a large pizza from *D.P. Dough*. You want 3 distinct toppings, chosen from their list of 11 vegetarian toppings.

(a) How many choices do you have for your pizza?
(b) How many choices do you have for your pizza if you refuse to have pineapple as one of your toppings?
(c) How many choices do you have for your pizza if you *insist* on having pineapple as one of your toppings?
(d) How do the three questions above relate to each other?