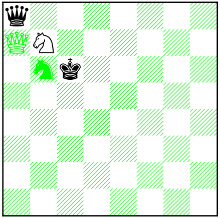
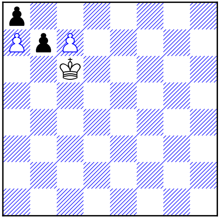




MA204/MA284 : Discrete Mathematics

Week 3: Binomials Coefficients

Dr Niall Madden

20 & 22 September 2017

These slides are based on §1.2 of Oscar Levin's *Discrete Mathematics: an open introduction*.

Tutorials (other than Tuesdays at 3) started this week. You should attend one of the sessions listed below.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			AdB-1020		
12 – 1		CA117			
1 – 2					
2 – 3	Tyndall		AC213		
3 – 4	IT202	*IT125*			
4 – 5					
5 – 6					

ASSIGNMENT 1 is now open!

To access the assignment, go to
http://mathswork.nuigalway.ie/webwork2/1718-MA284

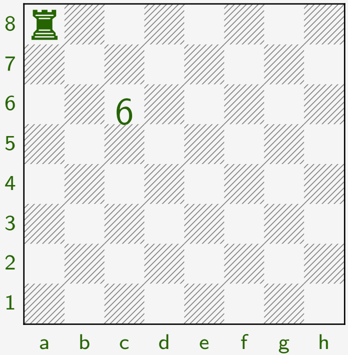
Your USERNAME is:

Your PASSWORD is:

There are 20 questions.
You may attempt each one up to 10 times.
This assignment contributes 10% to your final grade for Discrete Mathematics.

Deadline: 5pm, Thursday 6th October.

A rook can move only in straight lines (not diagonally). *Fill in each square of the chess board below with the number of different shortest paths the rook in the upper left corner can take to get to the square, moving one space at a time.* For example, there are **six** paths from the rook to the square **c6**: DDRR, DRDR, DRRD, RDDR, RDRD, and RRDD. (*R = right*, *D = down*).



A **bit** is a "binary digits" (i.e., 0 or 1).
A **bit string** is a string (list) of bits, e.g. 1001, 0, 111111, 10101010.

The *length* of the string is the number of bits.
A n-bit string has length n.
The set of all n-bit strings (for given n) is denoted $\mathbf{B}^n$.

Examples:

The *weight* of the string is the number of 1's.
The set of all n-bit strings of weight k is denoted $\mathbf{B}^n_k$.

Examples:

Bit strings

- The set of all n-bit strings (for given n) is denoted $\mathbf{B}^n$.
- The set of all n-bit strings of weight k is denoted $\mathbf{B}^n_k$.

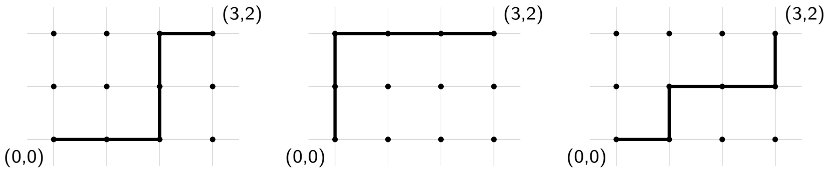
Some counting questions:

1. How many bit strings are there of length 5? That is, what is $|\mathbf{B}^5|$?
2. Of these, how many have weight 3? That is, what is $|\mathbf{B}^5_3|$?

The (integer) *lattice* is the set of all points in the Cartesian plane for which both the x and y coordinates are integers.

A *lattice path* is a **shortest possible path** connecting two points on the lattice, moving only horizontally and vertically.

Example: three possible lattice paths from the points $(0,0)$ to $(3,2)$ are:



Question: How many lattice paths are there from $(0,0)$ to $(3,2)$?

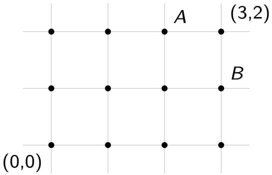
Useful observation 1

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as $|B_3^5|$.

Why?

Useful observation 2

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as the number from $(0,0)$ to $(2,2)$, plus the number from $(0,0)$ to $(3,1)$.



Version 1

What is the coefficient of (say) x^3y^2 in $(x+y)^5$?

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

So, by doing a lot of multiplication, we have worked out that the coefficient of x^3y^2 is 10 (which is rather familiar....)

But, not surprisingly there this a more systematic way of answering this problem.

Version 2

What is the coefficient of (say) x^3y^2 in $(x+y)^5$?

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y).$$

We can work out the coefficient of x^3y^2 in the expansion of $(x+y)^5$ by counting the number of ways we can choose three x's and two y's in

$$(x+y)(x+y)(x+y)(x+y)(x+y).$$

These numbers that occurred in all our examples are called *binomial coefficients*, and are denoted $\binom{n}{k}$

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as "} n \text{ choose } k\text{"}$$

1. $\binom{n}{k} = |\mathbf{B}_k^n|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n, each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
5. $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects.

If we were to skip ahead we would learn that there is a formula for

$$\binom{n}{k} \qquad \text{(that is, ``}n\text{ choose }k\text{'')}$$

that is expressed in terms of **factorials**.

Recall that the *factorial* of a natural number, n is

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1.$$

Examples:

We will eventually learn that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples

However, the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is not very useful in practice.

Example

There are exactly(!) 200 students in this Discrete Mathematics class. Of those, 17 are Arts students. How many other subsets of size 17 are there?

Answer: 1.8308×10^{24}. But this is not easy to compute...

Earlier, we learned that if the set of all n-bit strings with weight k is written $\mathbf{B}_k^n$, then

$$|\mathbf{B}_k^n| = |\mathbf{B}_{k-1}^{n-1}| + |\mathbf{B}_k^{n-1}|.$$

Similarly, we get find that...

Recurrence relation for $\binom{n}{k}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Why:

Recurrence relation for $\binom{n}{k}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This is often presented as *Pascal's Triangle*

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

Source: http://www.flickr.com/photos/35652310@N00/4139577452/.

Example

The NUIG Animal Shelter has 4 cats.

(a) How many choices do we have for a single cat to adopt?

(b) How many choices do we have if we want to adopt two cats?

(c) How many choices do we have if we want to adopt three cats?

(d) How many choices do we have if we want to adopt four cats?

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

Example: List all permutations of the letters `A`, `R` and `T`?

Important: order matters - "ART" $\neq$ "TAR" $\neq$ "RAT".

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

We can also count the number of permutations of the letters `A`, `R` and `T`, without listing them:

More generally, recall that $n!$ (read "n factorial") is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

E.g.,

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720.$$

$$10! = 3,628,800, \quad 20! = 2,432,902,008,176,640,000 \approx 2.43 \times 10^{18}.$$

There are $n!$ (i.e., n factorial) permutations of n (distinct) objects.

Q1. Let $S = \{1, 2, 3, 4, 5, 6\}$

(a) How many subsets are there total?
(b) How many subsets have $\{2, 3, 5\}$ as a subset?
(c) How many subsets contain at least one odd number?
(d) How many subsets contain exactly one even number?
(e) How many subsets are there of cardinality 4?
(f) How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?
(g) How many subsets of cardinality 4 contain at least one odd number?
(h) How many subsets of cardinality 4 contain exactly one even number?

Q2. How many subsets of $\{0, 1, \dots, 9\}$ have cardinality 6 or more? (Hint: Break the question into five cases).

Q3. How many shortest lattice paths start at (3,3) and end at (10,10)?
How many shortest lattice paths start at (3,3), end at (10,10), and pass through (5,7)?

Q4. Suppose you are ordering a large pizza from *D.P. Dough*. You want 3 distinct toppings, chosen from their list of 11 vegetarian toppings.

(a) How many choices do you have for your pizza?
(b) How many choices do you have for your pizza if you refuse to have pineapple as one of your toppings?
(c) How many choices do you have for your pizza if you *insist* on having pineapple as one of your toppings?
(d) How do the three questions above relate to each other?