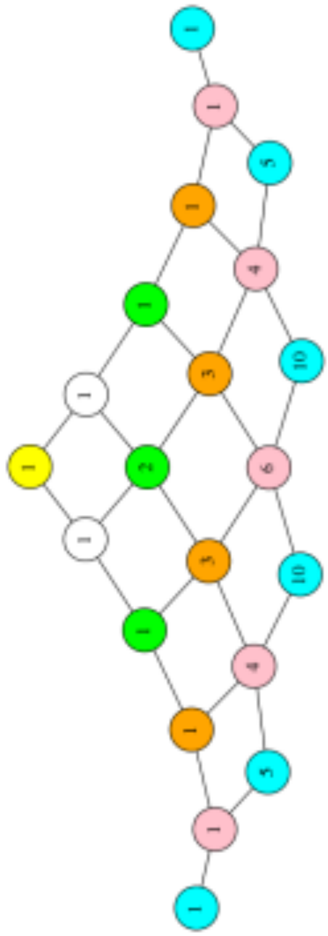
MA204/MA284 : Discrete Mathematics

Week 4: Permutations and Combinations

http://www.maths.nuigalway.ie/~niall/MA284/

27 and 29 September, 2017

These slides are based on §1.3 and §1.4 of Oscar Levin's *Discrete Mathematics: an open introduction*.



ASSIGNMENT 1 is now open!

To access the assignment, go to
`http://mathswork.nuigalway.ie/webwork2/1718-MA284`

Your USERNAME is: your ID number

Your PASSWORD is: your ID number

There are 20 questions.
You may attempt each one up to **20** times.
This assignment contributes 10% to your final grade for Discrete Mathematics.

Deadline: 5pm, Friday 6th October.

Tutorials

Some tutorials are not very well attended, particularly Wednesday at 11....

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			AdB-1020		
12 – 1		CA117			
1 – 2					
2 – 3	Tyndall		AC213		
3 – 4	IT202	IT125			
4 – 5					
5 – 6					

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as "} n \text{ choose } k \text{"}$$

1. $\binom{n}{k} = |\mathbf{B}_k^n|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
5. $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects.

There are two ways one can compute $\binom{n}{k}$

(a) the recurrence relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

(b) Using the formula, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

We justified the recurrence relation in (a) last week. We'll next derive (b), by relating it to the number of permutations of n objects.

Eg) How many subsets of $\{a, b, c, d\}$ have 2 elements.

(o) List them: $\{a,b\}$, $\{a,c\}$, $\{a,d\}$
$\{b,c\}$, $\{b,d\}$ $\{c,d\}$.

We count them to see $\binom{4}{2} = 6$.

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To apply (a): $\binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6$

To apply (b): $\dfrac{4!}{2!\,2!} = \dfrac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{(\cancel{2 \cdot 1})(2 \cdot 1)} = 6.$

Permutations

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

Number of permutations

There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

(i.e., n factorial) permutations of n (distinct) objects.

Example: The permutations of ABCD include

ABCD, ABDC, ADCB, DBCA,

ACBD, CBAD, etc. There are

$4 \times 3 \times 2 \times 1 = 24$ permutations in total.

In the previous example, we counted the number of permutations of n objects, where each permutation contained all n objects. Now consider a more general case.

Permutations of k objects from n

The number of permutations of k objects out of n is denoted $P(n,k)$. Its formula is

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Examples: How many 2 letter "words" can be made from 'ABCD',

We could list them all:

AB, AC, AD,
BA BC BD
CA CB CD
DA DB DC

4 choices for 1st letter (so 4 rows). And 3 choices for 2nd letter (so 3 cols).

Answer: $4 \times 3 = 12$

$= \frac{4!}{2!}$

Choosing the "backs" on a rugby team...

Suppose that the **the Ireland Rugby Team** has 5 backs: Hannah Tyrrell, Eimear Considine, Alison Miller, Mairead Coyne, and Niamh Briggs, all of whom can play Left Wing (11), Right Wing (14) and Full-back (15).

1. How many choices do we have for the left and right wingers (11 & 14)? (Note: picking Miller at 11 and Coyne at 14 is different from picking Coyne at 11 and Miller at 14.

Ans: 5 choices for 11 (left) then 4 for right

Total: $5 \times 4 = 20 = \frac{5!}{3!}$

2. How many choices do we have for the back 3 (11, 14 & 15)?

Ans: $5 \times 4 \times 3 = 60$.

Notice $P(n,k) = P(n,k-1) \cdot K$

Combinations (again)

A **combination** is a selection of objects, where order does not matter. That is, it is a **set**.

We have already seen that, if we have a set of n objects, there are $\binom{n}{k}$ subsets of size k.

So the number of **combinations** of k objects out of n is $\displaystyle\binom{n}{k}$.

Now we want to find a formula for $\displaystyle\binom{n}{k}$.

Choosing the "back 3" again...

Recall that our rugby team has 5 backs: Casey, McGinn, Miller, Baxter, and Briggs.

There are $\binom{5}{3}$ ways we can pick 3 of them for our team.

Once we have picked these three, there are $3! = 6$ ways we can assign them the Left wing, Right Wing and Full-back positions. That is

$$P(5,3) = \binom{5}{3} 3!.$$

However, we know $P(5,3)$, so this gives a formula for $\binom{5}{3}$.

ie

$$\binom{5}{3} = \frac{P(5,3)}{3!} = \frac{5!}{2!\,3!}$$

(1) We know there are $P(n,k)$ permutations of k objects out of n.

(2) We know that
$$P(n,k) = \frac{n!}{(n-k)!}$$

(3) Another way of making a permutation of k objects out of n is to

(a) Choose k from n without order. There are $\binom{n}{k}$ ways of doing this.

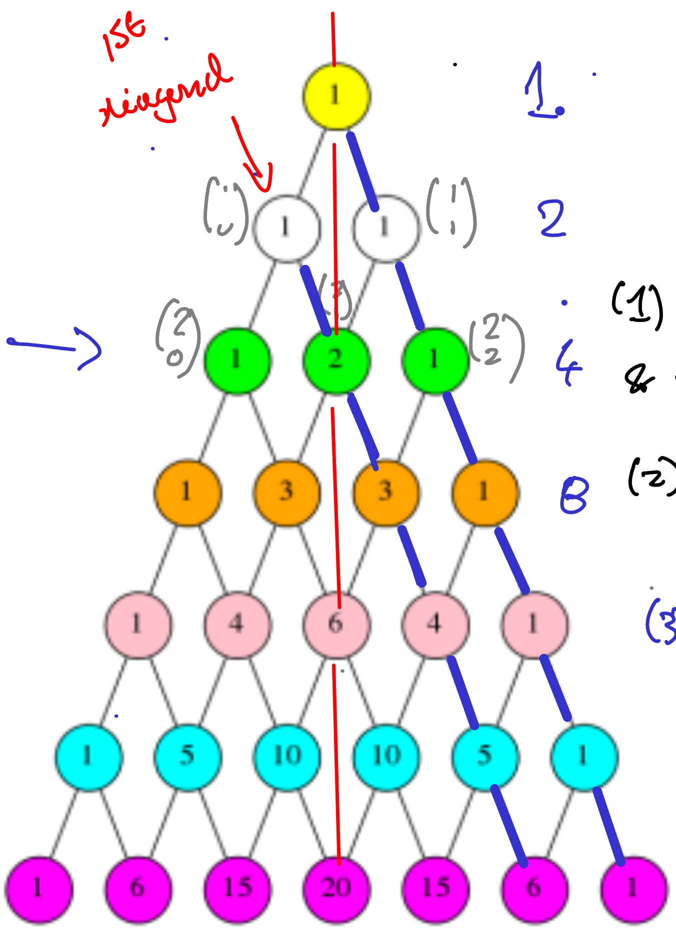
(b) Then count all the ways of ordering these k objects. There are $k!$ ways of doing this.

(c) By the Multiplicative Principle,
$$P(n,k) = \binom{n}{k} k!$$

(4) So now we know that $\dfrac{n!}{(n-k)!} = \dbinom{n}{k} k!$

(5) This gives the formula $\dbinom{n}{k} = \dfrac{n!}{(n-k)!k!}$

Finished here wednesday.



Binomial coefficients have many important properties.
Looking at their arrangement in Pascal's Triangle, several of these are obvious:

(1) All the numbers on the left & right edges are 1. $\binom{n}{0} = \binom{n}{n} = 1$

(2) First diagonal: $\{1, 2, 3, 4, 5 \ldots$
$= \{\binom{1}{0}, \binom{2}{1}, \binom{3}{2}, \ldots \binom{n}{n-1}\}$.

(3) Rows sum to a power of 2.
ie $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

(4) Each number is sum of 2 above: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.