MA204/MA284 : Discrete Mathematics

Week 4: Permutations and Combinations

http://www.maths.nuigalway.ie/~niall/MA284/

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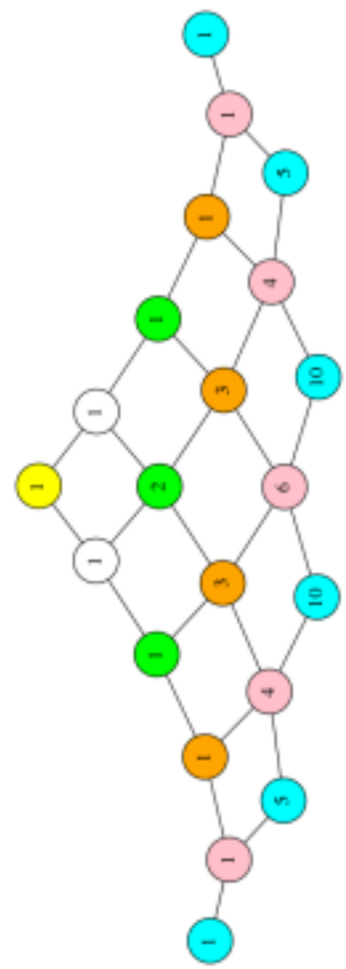
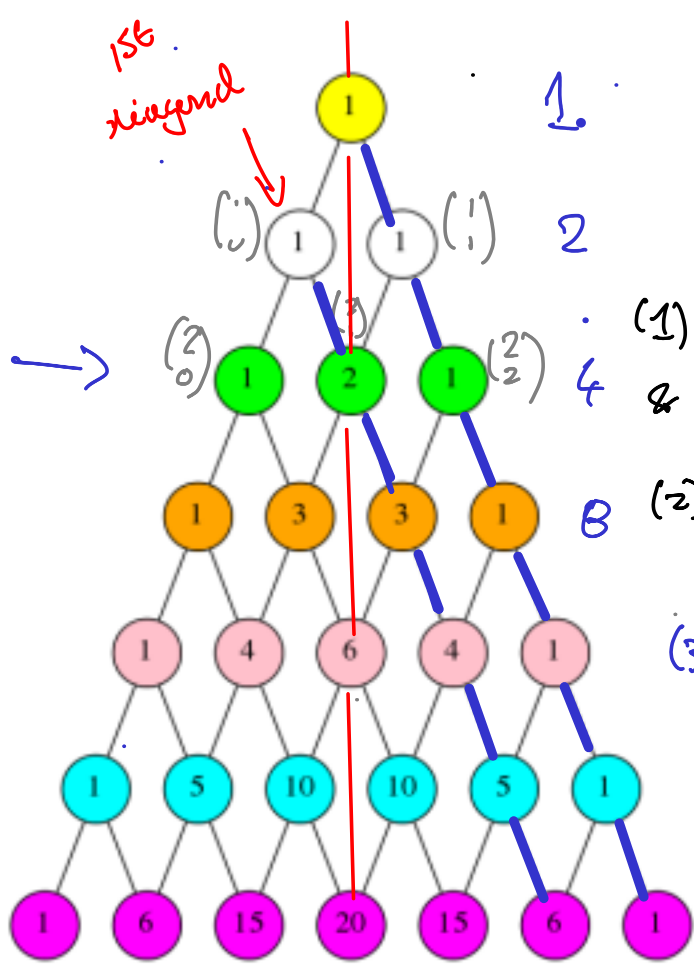
1. Recall...
   - ... Binomial coefficients
2. Permutations
3. Combinations, again
   - A formula
4. Algebraic and Combinatorial Proofs
5. Exercises

These slides are based on §1.3 and §1.4 of Oscar Levin's *Discrete Mathematics: an open introduction*.





Binomial coefficients have many important properties.
Looking at their arrangement in Pascal's Triangle, several of these are obvious:

(1) All the numbers on the left & right edges are 1. $\binom{n}{0} = \binom{n}{n} = 1$

(2) First diagonal: $\{1, 2, 3, 4, 5 \ldots$ $= \{\binom{1}{0}, \binom{2}{1}, \binom{3}{2}, \ldots \binom{n}{n-1}\}$.

(3) Rows sum to a power of 2.
ie $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

(4) Each number is sum of 2 above: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Proofs

Proofs of identities involving Binomial coefficients can be classified as

- **Algebraic:** if they rely mainly on the formula for binomial coefficients.
- **Combinatorical:** if the involve counting a set in two different ways.

For our first example, we will give two proofs of the following fact:

$$\overset{A}{\binom{n}{k}} = \overset{B}{\binom{n}{n-k}}.$$

Algebraic: Let $A = \binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

Let $B = \binom{n}{n-k} = \dfrac{n!}{(n-k)!\,(n-(n-k))!} = \dfrac{n!}{k!\,(n-k)!} = A$

So $A = B$

Algebraic proof of Pascal's triangle recurrence relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Let $A = \binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

Let $B = \binom{n-1}{k-1} + \binom{n-1}{k} = \dfrac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} + \dfrac{(n-1)!}{k!\,(n-k-1)!}$

$$= \frac{(n-1)!}{(k-1)!\,(n-k)!} \times \frac{k}{k} + \frac{(n-1)!}{k!\,(n-k-1)!}\,\frac{n-k}{n-k}$$

$$= \frac{(n-1)!\,(k)}{k!\,(n-k)!} + \frac{(n-1)!\,(n-k)}{k!\,(n-k)!} = \frac{(n-1)!\,(\cancel{k}+n-\cancel{k})}{k!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!}$$

So $A = B$.

Combinatorial proof of Pascal's triangle recurrence relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

See OHP.

Algebraic and Combinatorial Proofs

Which are better: algebraic or combinatorial proofs?

When we first study discrete mathematics, *algebraic* proofs make seem easiest: they reply only on using some standard formulae, and don't require any deeper insight. Also, they are more "familiar".

However,

- Often algebraic proofs are quite tricky;
- Usually, algebraic proofs give no insight as to why a fact is true.

Example

Give a combinatorial proof of the following fact

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Check, $n=2$. $\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = 1^2 + 2^2 + 1^2 = 6$ $\binom{4}{2} = \frac{4!}{2!2!} = 6.$

We wish to show that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

Let A & B be sets with n distinct objects.

That is $|A| = n$, $|B| = n$, $|A \cap B| = 0$ (so $|A \cup B| = 2n$).

So the number of subsets of size n ~~of~~ of $A \cup B$ is $\binom{2n}{n}$

Alternatively, we could choose

0 from A & n from B in $\binom{n}{0}\binom{n}{n}$ ways, OR

1 from A & $n-1$ from B in $\binom{n}{1}\binom{n}{n-1}$ " OR, etc,

⋮

n from A & 0 from B in $\binom{n}{n}\binom{n}{0}$ ways.

So $\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}$.

But $\binom{n}{k} = \binom{n}{n-k}$ so $\binom{n}{n} = \binom{n}{0}$, $\binom{n}{n-1} = \binom{n}{1}$, etc, giving

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Algebraic and Combinatorial Proofs

Example

Give two proofs of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

First, we check:

$n=1$ $\binom{1}{0}+\binom{1}{1} = 1+1 = 2^1$ ✓

$n=2$ $\binom{2}{0}+\binom{2}{1}+\binom{2}{2} = 1+2+1 = 4 = 2^2$ ✓

$n=3$ $\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3} = 1+3+3+1 = 2^3$ ✓

Algebraic proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Recall that $\binom{n}{k}$ is the coef of $x^k y^{n-k}$ in $(x+y)^n$. So

$$(x+y)^n = \binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{n} x^n y^0$$

Now take $x=1$ & $y=1$. So

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

Combinatorial proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Recall that B^n is the set of bit strings of lenght n & there are 2^n of these.

B^n_k is the set of bit strings of weight k. There are $\binom{n}{k}$ of these.

Also,

$$B^n = B^n_0 \cup B^n_1 \cup B^n_2 \cup \ldots \cup B^n_n$$

So

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

Finished here but annotated versions of remaining slides posted on-line.

Algebraic and Combinatorial Proofs

What is a "combinatorial proof" really?

1. These proofs involve finding two different ways to answer the same counting question.
2. Then we explain why the answer to the problem posed one way is A
3. Next we explain why the answer to the problem posed the other way is B.
4. Since A and B are answers to the same question, we have shown it must be that $A = B$.

Next two slides give outline an example not discussed in class.

Example

MA204 Semester 1 Examination 2014/15: Q1(c) Using a combinatorial argument, or otherwise, prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof 1: (Algebraic)

Let $A = k\binom{n}{k}$. Since $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$, we get

$A = \frac{k(n!)}{k!\,(n-k)!} = \frac{n!}{(k-1)!\,(n-k)!}$

Next Let $B = n\binom{n-1}{k-1}$. So $B = \frac{n\,(n-1)!}{(k-1)!\,(n-1-(k-1))!}$

$= \frac{n!}{(k-1)!\,(n-k)!}$. So $A = B$ as required.

Example

MA204 Semester 1 Examination 2014/15: Q1(c) Using a combinatorial argument, or otherwise, prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof 2: (Combinatorical). Suppose we wanted to pick k people from n to play on a soccer team, this could be done in $\binom{n}{k}$ ways. Then we could designate one of those k to be the 'keeper. This can be done in k ways. So, in total, there are $k\binom{n}{k}$ choices.

Alternatively, we could pick the 'keeper first, in n ways, and then choose the remaining $k-1$ players from the other $n-1$ people, in $\binom{n-1}{k-1}$ ways. So thats $n\binom{n-1}{k-1}$. But the two approaches give the same outcomes, so $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Exercises

All these are taken from Section 1.4 of Levin's *Discrete Mathematics*.

Q1. Give a combinatorial proof for the identity $1+2+3+\cdots+n=\binom{n+1}{2}$.

Q2. Give an algebraic proof, using induction, for the identity
$1+2+3+\cdots+n=\binom{n+1}{2}$.

Q3. Give a combinatorial proof of the fact that $\binom{x+y}{2}-\binom{x}{2}-\binom{y}{2}=xy$

Q4. Give a combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k-2}=\binom{n}{k}\binom{k}{2}$.

Q5. Consider the bit strings in $\mathbf{B}^6_2$ (bit strings of length 6 and weight 2).

(a) How many of those bit strings start with 01?
(b) How many of those bit strings start with 001?
(c) Are there any other strings we have not counted yet? Which ones, and how many are there?
(d) How many bit strings are there total in $\mathbf{B}^6_2$?
(e) What binomial identity have you just given a combinatorial proof for?

Q6. Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2}+\binom{3}{2}\binom{n-1}{2}+\binom{4}{2}\binom{n-2}{2}+\cdots+\binom{n}{2}\binom{2}{2}=\binom{n+3}{5}.$$