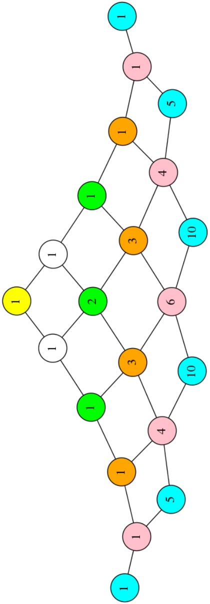
MA204/MA284 : Discrete Mathematics

Week 4: Permutations and Combinations

http://www.maths.nuigalway.ie/~niall/MA284/

27 and 29 September, 2017



1 Recall...
   - ... Binomial coefficients
2 Permutations
3 Combinations, again
   - A formula
4 Algebraic and Combinatorial Proofs
5 Exercises

These slides are based on §1.3 and §1.4 of Oscar Levin's *Discrete Mathematics: an open introduction*.

ASSIGNMENT 1 is now open!

To access the assignment, go to
http://mathswork.nuigalway.ie/webwork2/1718-MA284

Your USERNAME is:

Your PASSWORD is:

There are 20 questions.
You may attempt each one up to **20** times.
This assignment contributes 10% to your final grade for Discrete Mathematics.

Deadline: 5pm, Friday 6th October.

Tutorials

Some tutorials are not very well attended, particularly Wednesday at 11....

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			AdB-1020		
12 – 1		CA117			
1 – 2					
2 – 3	Tyndall		AC213		
3 – 4	IT202	IT125			
4 – 5					
5 – 6					

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as ``} n \text{ choose } k\text{''}$$

1. $\binom{n}{k} = |\mathbf{B}_k^n|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
5. $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects.

There are two ways one can compute $\binom{n}{k}$

(a) the recurrence relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

(b) Using the formula, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

We justified the recurrence relation in (a) last week. We'll next derive (b), by relating it to the number of permutations of n objects.

Permutations

A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

Number of permutations

There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

(i.e., n factorial) permutations of n (distinct) objects.

Example:

Permutations

In the previous example, we counted the number of permutations of n objects, where each permutation contained all n objects. Now consider a more general case.

Permutations of k objects from n

The number of permutations of k objects out of n is denoted $P(n, k)$. Its formula is

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Examples:

Choosing the "backs" on a rugby team...

Suppose that the **the Ireland Rugby Team** has 5 backs: Hannah Tyrrell, Eimear Considine, Alison Miller, Mairead Coyne, and Niamh Briggs, all of whom can play Left Wing (11), Right Wing (14) and Full-back (15).

1. How many choices do we have for the left and right wingers (11 & 14)? (Note: picking Miller at 11 and Coyne at 14 is different from picking Coyne at 11 and Miller at 14.

2. How many choices do we have for the back 3 (11, 14 & 15)?

Combinations (again)

A **combination** is a selection of objects, where order does not matter. That is, it is a **set**.

We have already seen that, if we have a set of n objects, there are $\binom{n}{k}$ subsets of size k.

So the number of **combinations** of k objects out of n is $\displaystyle\binom{n}{k}$.

Now we want to find a formula for $\displaystyle\binom{n}{k}$.

Choosing the "back 3" again...

Recall that our rugby team has 5 backs: Casey, McGinn, Miller, Baxter, and Briggs.

There are $\binom{5}{3}$ ways we can pick 3 of them for our team.

Once we have picked these three, there are $3! = 6$ ways we can assign them the Left wing, Right Wing and Full-back positions. That is

$$P(5,3) = \binom{5}{3} 3!.$$

However, we know $P(5,3)$, so this gives a formula for $\binom{5}{3}$.

(1) We know there are $P(n,k)$ permutations of k objects out of n.

(2) We know that

$$P(n,k) = \frac{n!}{(n-k)!}$$

(3) Another way of making a permutation of k objects out of n is to

(a) Choose k from n without order. There are $\binom{n}{k}$ ways of doing this.

(b) Then count all the ways of ordering these k objects. There are $k!$ ways of doing this.

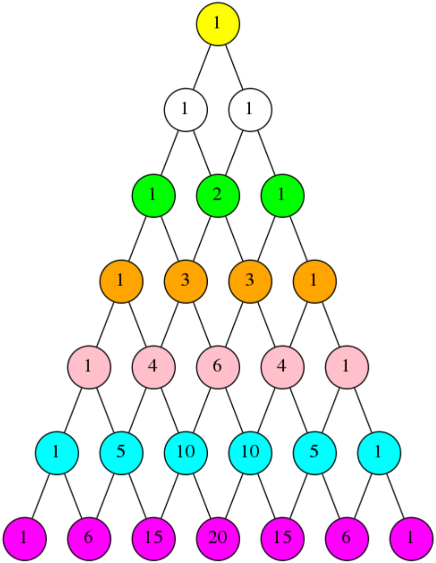
(c) By the Multiplicative Principle,

$$P(n,k) = \binom{n}{k} k!$$

(4) So now we know that $\dfrac{n!}{(n-k)!} = \dbinom{n}{k} k!$

(5) This gives the formula $\dbinom{n}{k} = \dfrac{n!}{(n-k)!k!}$

Algebraic and Combinatorial Proofs



Binomial coefficients have many important properties.
Looking at their arrangement in Pascal's Triangle, several of these are obvious:

Proofs

Proofs of identities involving Binomial coefficients can be classified as

- **Algebraic:** if they rely mainly on the formula for binomial coefficients.
- **Combinatorical:** if the involve counting a set in two different ways.

For our first example, we will give two proofs of the following fact:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Algebraic proof of Pascal's triangle recurrence relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Combinatorial proof of Pascal's triangle recurrence relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Which are better: algebraic or combinatorial proofs?

When we first study discrete mathematics, *algebraic* proofs make seem easiest: they reply only on using some standard formulae, and don't require any deeper insight. Also, they are more "familiar".

However,

- Often algebraic proofs are quite tricky;
- Usually, algebraic proofs give no insight as to why a fact is true.

Example

Give a combinatorial proof of the following fact

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

We wish to show that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

Example

Give two proofs of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

First, we check:

Algebraic proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Combinatorial proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

What is a "combinatorial proof" really?

1. These proofs involve finding two different ways to answer the same counting question.
2. Then we explain why the answer to the problem posed one way is A
3. Next we explain why the answer to the problem posed the other way is B.
4. Since A and B are answers to the same question, we have shown it must be that $A = B$.

Example

MA204 Semester 1 Examination 2014/15: Q1(c) Using a combinatorial argument, or otherwise, prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof 1:

Example

MA204 Semester 1 Examination 2014/15: Q1(c) Using a combinatorial argument, or otherwise, prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof 2:

Exercises

All these are taken from Section 1.4 of Levin's *Discrete Mathematics*.

Q1. Give a combinatorial proof for the identity $1+2+3+\cdots+n=\binom{n+1}{2}$.

Q2. Give an algebraic proof, using induction, for the identity $1+2+3+\cdots+n=\binom{n+1}{2}$.

Q3. Give a combinatorial proof of the fact that $\binom{x+y}{2}-\binom{x}{2}-\binom{y}{2}=xy$

Q4. Give a combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k-2}=\binom{n}{k}\binom{k}{2}$.

Q5. Consider the bit strings in $\mathbf{B}_2^6$ (bit strings of length 6 and weight 2).

(a) How many of those bit strings start with 01?

(b) How many of those bit strings start with 001?

(c) Are there any other strings we have not counted yet? Which ones, and how many are there?

(d) How many bit strings are there total in $\mathbf{B}_2^6$?

(e) What binomial identity have you just given a combinatorial proof for?

Q6. Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2}+\binom{3}{2}\binom{n-1}{2}+\binom{4}{2}\binom{n-2}{2}+\cdots+\binom{n}{2}\binom{2}{2}=\binom{n+3}{5}.$$