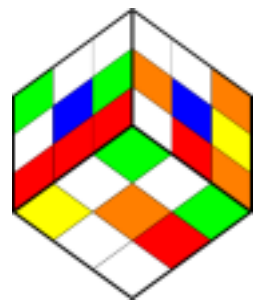
Friday's notes

MA204/MA284 : Discrete Mathematics

Week 5: Stars and Bars

http://www.maths.nuigalway.ie/~niall/MA284/

4 and 6 October, 2017



These slides are based on Sections 1.4 and 1.5 of Oscar Levin's *Discrete Mathematics: an open introduction*.

NNI

A non-negative integer problem

How many non-negative integer solutions are there to the problem

$$x_1 + x_2 + \cdots + x_k = n?$$

This is the same as...

How many ways are there to distribute n identical objects among k individuals.

The answer is $\binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!(k-1)!}$ $= \binom{n+k-1}{n}$

That is x_1 is the number of objects given to peson 1,
x_2 is the number given to Person 2,
etc.

≥ 0

Example (Part 1)

1. How many non-negative integer solutions are there to $x_1 + x_2 + x_3 = 3$?

note that $x_1 = 4$, $x_2 = 1$ & $x_3 = -2$ is not a solution.

Neither is $x_1 = 1/2$ $x_2 = 2$ $x_3 = 1/2$. ← not ints

Solutions are

x_1	0	0	0	0	1	1	1	2	2	3
x_2	0	1	2	3	0	1	2	0	1	0
x_3	3	2	1	0	2	1	0	1	0	0

There are 10 solutions in total.

here $n=3$, $k=3$

So there are

$$\binom{5}{2} = \frac{5!}{2!\,3!}$$

$= 10$ solutions.

Example (Part 1)

1. How many non-negative integer solutions are there to $x_1 + x_2 + x_3 = 3$?

Solutions are

	(i)				(v)					
x_1	0	0	0	0	1	1	1	2	2	3
x_2	0	1	2	3	0	1	2	0	1	0
x_3	3	2	1	0	2	1	0	1	0	0

In "stars & bars"

(i) $x_1 = 0,\ x_2 = 0,\ x_3 = 3$ is $|\ |\ *\ *\ *$

(v) $x_1 = 1,\ x_2 = 0,\ x_3 = 2$ is $*\ |\ |\ *\ *$

Example (Part 2)

2. How many non-negative integer solutions are there to $x_1 + x_2 \leq 3$?

we list all the solutions

x_1	0	0	0	0	1	1	1	2	2	3	~~4~~
x_2	0	1	2	3	0	1	2	0	1	0	

There are 10 possible solutions.

This is exactly the same as for $x_1+x_2+x_3=3$ but with "an invisible x_3".

Looking at this example, it seems that

The number of non-negative integer solutions to

$$(1) \qquad x_1 + x_2 + x_3 + \cdots + x_k + x_{k+1} = n.$$

is the same as the number of non-negative integer solutions to

$$(2) \qquad x_1 + x_2 + x_3 + \cdots + x_k \leq n,$$

which is the same as the number of non-negative integer solutions to

$$(3) \qquad x_1 + x_2 + x_3 + \cdots + x_k < n + 1,$$

Why is that?

If we have a solution to (1) then subtract x_{k+1} from both sides. The LHS becomes $x_1 + x_2 + \cdots + x_k$ & the RHS becomes $\leq n$ since $x_{k+1} \geq 0$.

Looking at this example, it seems that

The number of non-negative integer solutions to

$$(1) \qquad x_1 + x_2 + x_3 + \cdots + x_k + x_{k+1} = n.$$

is the same as the number of non-negative integer solutions to

$$(2) \qquad x_1 + x_2 + x_3 + \cdots + x_k \leq n,$$

which is the same as the number of non-negative integer solutions to

$$(3) \qquad x_1 + x_2 + x_3 + \cdots + x_k < n + 1,$$

Exer: show $(2) \Longleftrightarrow (3)$

Why is that?

Simlarly, suppose we have a solution to (2). Then set $x_{k+1} = n - x_1 - x_2 - \cdots - x_k$. note that $x_{k+1} \geq 0$. Then add to both sides of (2) to get a solution to (1).

Example (MA284, Semester 1 Exam, 2015/16, Q5(b))

(i) How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13$$

(ii) How many non-negative integer solutions are there of the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 13$$

(i) This is the same as distributing $n=13$ objects among $k=5$. So there are $\binom{n+k-1}{k-1} = \binom{17}{4} = 2380$.

(ii) This is the same as $x_1 + x_2 + \cdots + x_5 + X_6 = 13$

Here $n=13$, $k=6$, so there are $\binom{18}{5} = 8568$

In the next section, we will study counting the number of non-negative integer solutions to problems, where there is a upper bound on solutions.

To lead into that, here is a variant on a problem from earlier.

Example

(a) In how many ways can one distribute **eight** €1 coins to **three** students so that each student receives **at least** €1?

(b) In how many ways can one distribute **eight** €1 coins to **three** students so that each student receives *at most* €3?

(a) First give €1 to each, and give the remaining 5 to them in $\binom{5+3-1}{3-1} = \binom{7}{2} = 21$.

(b) We could count all solutions to $x_1 + x_2 + x_3 = 8$ and then count all solutions with $x_1 > 3$, $x_2 > 3$ & $x_3 > 3$, and then subtract that. Need PIE.

Recall that if we have sets A and B, then

- $|A|$ and $|B|$ denote the number of elements in A and B, respectively.
- The *union of A and B* is the set of all elements in either A or B. We write it as $A \cup B$.
- The *intersection of A and B* is the set of all elements found in *both* A and B. We write it as $A \cap B$.

The *Principle of Inclusion/Exclusion (PIE)* for two sets, A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example: $A = \{n, u, i, g\}$ $B = \{g, m, i, t\}$

$A \cup B = \{g, i, m, n, t, u\}$ $|A \cup B| = 6$

$A \cap B = \{g, i\}$ $|A \cap B| = 2$. So

$|A \cup B| = 4 + 4 - 2 = 6$ ✓