MA204/MA284 : Discrete Mathematics

Week 5: Stars and Bars

http://www.maths.nuigalway.ie/~niall/MA284/

4 and 6 October, 2017

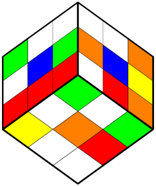
1 Stars and bars
- An "Investigate" activity
- 7 apples for 4 people
- Multisets

2 Problems with non-negative integer solutions
- Inequalities

3 NNI equations with lower bounds on solutions

4 Advanced Counting Using PIE

5 Exercises



These slides are based on Sections 1.4 and 1.5 of Oscar Levin's *Discrete Mathematics: an open introduction*.

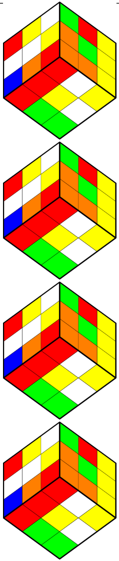
1. Assignment 1 is due 5pm, Friday 6 Oct 2017
   To access the assignment, go to
   http://mathswork.nuigalway.ie/webwork2/1718-MA284

2. Friday's lecture will take place in the Tyndall lecture theatre (because AM200 will be in use for Open Day).

Suppose you have some number of identical Rubik's cubes to distribute to your friends. Start by creating a single row of the cubes. Now find the number of different ways you can distribute the cubes provided:

1. you have 3 cubes to give to 2 people.
2. you have 4 cubes to give to 2 people.
3. you have 5 cubes to give to 2 people.
4. you have 3 cubes to give to 3 people.
5. you have 4 cubes to give to 3 people.
6. you have 5 cubes to give to 3 people.

Make a conjecture about how many different ways you could distribute 7 cubes to 4 people. Explain.
What if each person were required to get *at least one* cube? How would your answers change?



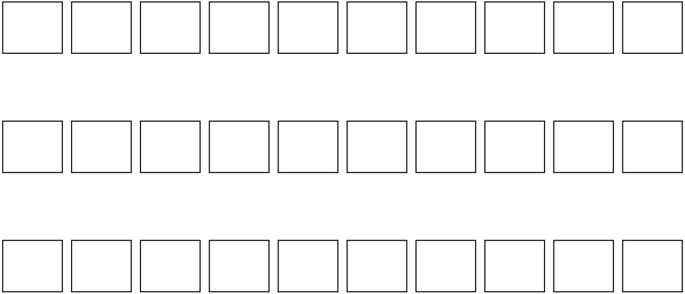
Think about this question during this lecture...

Every day you give some apples to your lecturers. Today you have **7** apples. How many ways can you give them to the **4** lecturers you have today?

Every day you give some apples to your lecturers. Today you have **7** apples. How many ways can you give them to the **4** lecturers you have today?

One can represent any solution by filling out 10 boxes with 7 stars and 3 bars.

Examples:



Every day you give some apples to your lecturers. Today you have **7** apples. How many ways can you give them to the **4** lecturers you have today?

- *Every solution can be represented by 10 boxes, each with a star or a bar.*
- There are 7 stars and 3 bars in total.
- We can choose any 3 of the 10 boxes in which to place the bars, and then put the stars in the rest.
- **So we have $\binom{10}{3}$ choices for where to put the bars**.

Definition (Multiset)

A *multiset* is a set of objects, where each object can appear more than once. As with an ordinary set, order does not matter.

Examples:

How many *multisets* of size 4 can you form using numbers $\{1, 2, 3, 4, 5\}$?

How many *multisets* of size n can you form using the numbers $\{1, 2, 3, \ldots, k\}$?

Example

MA204 Semester 1 Examination 2014/15: Q2(a)

1. In how many ways can one distribute **ten** €1 coins to four students?
2. In how many ways can one distribute **ten** €1 coins to four students so that each student receives at least €1?

A non-negative integer problem

How many non-negative integer solutions are there to the problem

$$x_1 + x_2 + \cdots + x_k = n?$$

This is the same as...

How many ways are there to distribute n identical objects among k individuals.

The answer is $\binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!(k-1)!}$

Example (Part 1)

1. How many non-negative integer solutions are there to $x_1 + x_2 + x_3 = 3$?

Example (Part 2)

2. How many non-negative integer solutions are there to $x_1 + x_2 \leq 3$?

Looking at this example, it seems that

The number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + \cdots + x_k + x_{k+1} = n.$$

is the same as the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + \cdots + x_k \leq n,$$

which is the same as the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + \cdots + x_k < n + 1,$$

Why is that?

Example (MA284, Semester 1 Exam, 2015/16, Q5(b))

(i) How many non-negative integer solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13$$

(ii) How many non-negative integer solutions are there of the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 13$$

In the next section, we will study counting the number of non-negative integer solutions to problems, where there is a upper bound on solutions.

To lead into that, here is a variant on a problem from earlier.

Example

(a) In how many ways can one distribute **eight** €1 coins to **three** students so that each student receives **at least** €1?

(b) In how many ways can one distribute **eight** €1 coins to **three** students so that each student receives *at most* €3?

Recall that if we have sets A and B, then

- $|A|$ and $|B|$ denote the number of elements in A and B, respectively.
- The *union of A and B* is the set of all elements in either A or B. We write it as $A \cup B$.
- The *intersection of A and B* is the set of all elements found in *both* A and B. We write it as $A \cap B$.

The *Principle of Inclusion/Exclusion (PIE)* for two sets, A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example:

The *Principle of Inclusion/Exclusion (PIE)* for three sets, A, B and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example:

We will now use this idea to solve non-negative integer problems, with *upper bounds* on the solution.

Example

Students work together in groups of 3 on a Discrete Mathematics assignment. The group is given a score, which they divide up (according to the amount of work each did) to get their individual scores. The maximum individual score is 4 marks.

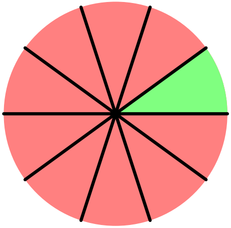
Aoife, Brian and Conor worked together as a group. Their group got a score of **11**. How many ways can this be divided out?

The PIE works for larger numbers of sets too, although it gets a little messy to write down. For **4** sets, we can think of it as

$$\begin{aligned} |A \cup B \cup C \cup D| = &\text{(the sum of the sizes of each single set)} \\ &- \text{(the sum of the sizes of each union of 2 sets)} \\ &+ \text{(the sum of the sizes of each union of 3 sets)} \\ &- \text{(the sum of the sizes of union of all 4 sets)} \end{aligned}$$

Example (Example 1.6.2 of text-book)

How many ways can we distribute 10 slices of pie(!) to 4 kids so that no kid gets more than 2 slices?



Not all such problems have trivial solutions.

Example

How many non-negative integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13$$

if $0 \le x_i \le 3$ for each i.

Unless indicated otherwise, these questions identical to, or variants on, Sections 1.5 and 1.6 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. A *multiset* is a collection of objects, just like a set, but can contain an object more than once (the order of the elements still doesn't matter). For example, $\{1, 1, 2, 5, 5, 7\}$ is a multiset of size 6.

(a) How many *sets* of size 5 can be made using the 10 digits: 0, 1, . . . 9?
(b) How many *multi*sets of size 5 can be made using the 10 digits: 0, 1, . . . 9?

Q2. Each of the counting problems below can be solved with stars and bars. For each, say what outcome the diagram $***|*||**|$ represents, if there are the correct number of stars and bars for the problem. Otherwise, say why the diagram does not represent any outcome, and what a correct diagram would look like.

(a) How many ways are there to select a handful of 6 jellybeans from a jar that contains 5 different flavors?
(b) How many ways can you distribute 5 identical lollipops to 6 kids?
(c) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 6$.

Q3. (a) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition of letters?"
(b) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition, if the letters must be in alphabetical order?"

Q4. After gym class you are tasked with putting the 14 identical balls away into 5 bins.

(a) How many ways can you do this if there are no restrictions?
(b) How many ways can you do this if each bin must contain at least one ball?
(c) How many ways can you do this if no bin can hold more than 6 balls?