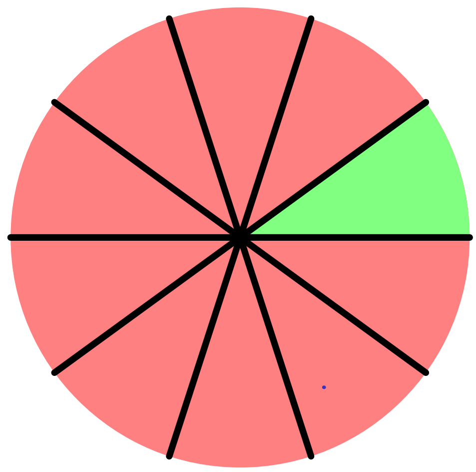
MA284 : Discrete Mathematics

Week 6: Advance applications of the PIE

http://www.maths.nuigalway.ie/~niall/MA284

11 and 13 of October, 2017



See also §1.6 of Levin's *Discrete Mathematics: an open introduction*.

Assignment 1 is now closed

Your grades are available from the Blackboard Grade Centre.
The average score was ... 18·5 (out of 20).

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Assignment 2 is now open!

To access the assignment, go to
`http://mathswork.nuigalway.ie/webwork2/1718-MA284`

Your USERNAME and PASSWORD are: your ID number.
There are **20** questions. You may attempt each one up to **10** times.
This assignment contributes 10% to your final grade for Discrete Mathematics.

Deadline: 5pm, Friday, 27 October.

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Please complete the online survey on MA284 to provide us with feedback on improving it. The link is in an announcement on Blackboard.

Last week we had the following question

How many ways can you share **n** apples among your **k** lecturers?

- This is the same as finding the number of ways we can arrange n apples (stars), divided into k groups, separated by $k-1$ bars.
- Any way can be written with $n+k-i$ symbols (n stars and $k-1$ bars): we just have to choose where to put the $k-1$ bars. This can be done in $\binom{n+k-1}{k-1} = \binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(k-1)!}$ ways.
- This can also be thought of as a *multiset*: a set of objects, where each object can appear more than once.
- And it can be framed as the number of solutions to the **non-negative integer problem:**

$$x_1 + x_2 + \cdots + x_k = n.$$

These three problems have the same number of solutions

The number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + \cdots + x_k + x_{k+1} = n. \tag{1}$$

is the same as the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + \cdots + x_k \leq n, \tag{2}$$

which is the same as the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + \cdots + x_k < n + 1, \tag{3}$$

We ended with solving variants on these problems that have constraints, for example

(a) where each of the x_k is at least 2; (easy)

(b) where each of the x_k is at most 5. (harder).

Solving the latter brought us back to the PIE.

Recall that

- $|X|$ denotes the number of elements in the set X.
- $X \cup Y$ (the *union of X and Y*) is the set of all elements that belong to **either** X or Y.
- $A \cap B$ (the *intersection of X and Y*) is the set of all elements that belong to *both* X and Y.

The **Principle of Inclusion/Exclusion (PIE)** for two sets, A and B, is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For for three sets, A, B and C, the PIE is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example:

We will now use this idea to solve non-negative integer problems, with *upper bounds* on the solution.

Example

Students work together in groups of 3 on a Discrete Mathematics assignment. The group is given a score, which they divide up (according to the amount of work each did) to get their individual scores. The maximum individual score is 4 marks.
Aoife, Brian and Conor worked together as a group. Their group got a score of **11**. How many ways can this be divided out?

Eg, Aoife gets 4, Brian gets 4, Conor gets 3.
Total is 11.

But not 3 for Aoife, 3 for Brian & 5 for Conor.
To solve this, let A be the set of outcomes where Aoife gets 5 or more, B the set where Brian gets 5 or more...

We will now use this idea to solve non-negative integer problems, with *upper bounds* on the solution.

Example

Students work together in groups of 3 on a Discrete Mathematics assignment. The group is given a score, which they divide up (according to the amount of work each did) to get their individual scores. The maximum individual score is 4 marks.
Aoife, Brian and Conor worked together as a group. Their group got a score of **11**. How many ways can this be divided out?

So (see OHP) $|A| = |B| = |C| = 28$.
Next calutate $|A \cap B|$ ie number of outcomes where both Aoife & Conor get 5 or more. This is equal to the number of solutions to $x_1 + x_2 + x_3 = 1$

There are $\binom{3}{2} = 3$ of these. Similarly $|A \cap C| = |B \cap C| = 3$.
Finally $|A \cap B \cap C| = 0$. (See OHP for rest)

The PIE works for larger numbers of sets too, although it gets a little messy to write down. For **4** sets, we can think of it as

$|A \cup B \cup C \cup D|$ = (the sum of the sizes of each single set)

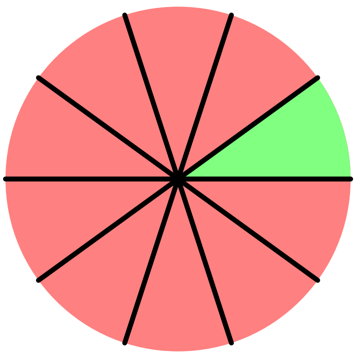
− (the sum of the sizes of each ~~union~~ intersection of 2 sets)

+ (the sum of the sizes of each ~~union~~ intersection of 3 sets)

− (the sum of the sizes of ~~union~~ intersection. of all 4 sets)

Example (Example 1.6.2 of text-book)

How many ways can we distribute 10 slices of pie(!) to 4 kids so that no kid gets more than 2 slices?



See text book

(Hint : zero !).

Not all such problems have easy solution solutions.

Example

How many non-negative integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...}$$

1. There are no restrictions (other than each x_i being an *nni*).
2. $0 \leq x_i \leq 3$ for each i.

1. Here $n=13$ and $k=5$, so there are $\binom{n+k-1}{k-1} = \binom{17}{4} = 2380$ solutions.

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2. If one of the x_i is $x_i \geq 4$, this is the same as the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 9$, which is $\binom{13}{4} = 715$.

If two of the x_i are greater than 3, solve $x_1 + x_2 + x_3 + x_4 + x_5 = 5$.

This has $\binom{9}{4} = 126$ solutions.

If 3 of the $x_i \geq 4$, Solve $x_1 + x_2 + x_3 + x_4 + x_5 = 1$ in $\binom{5}{4} = 5$ ways.

For **Assignment 2** of MA840 (*Indiscreet Mathematics*), students work together in groups of **4**. The group is given a score, which they divide up, according to the amount of work each did, to get their individual scores.
Aoife, Brian, Conor and Dana worked together, and got a score of **10**. They decided it should be divided as:

Aoife: 1 Brian: 2 Conor: 3 Dana: 4.

They informed their lecturer of this, and he tried to enter these on Blackboard. But he is not very good with computers, and got ALL the scores wrong!
How many ways could this happen?

Eg we should have $\{a, b, c, d\} = \{1, 2, 3, 4\}$.
But $a=4$, $b=3$, $c=2$ & $d=1$ is a "derangement"
as is $a=2$, $b=1$, $c=4$, $d=3$.
But not $a=2$, $b=3$, $c=1$, $d=4$ because $d=4$ is correct.

What are all the possible derangements?