Friday's notes

MA284 : Discrete Mathematics

Week 6: Advance applications of the PIE

http://www.maths.nuigalway.ie/~niall/MA284

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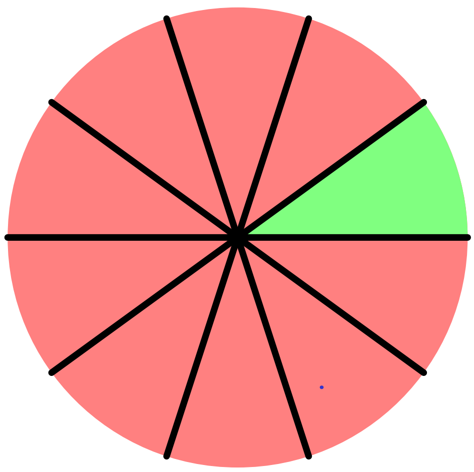

See also §1.6 of Levin's *Discrete Mathematics: an open introduction*.

For **Assignment 2** of MA840 (*Indiscreet Mathematics*), students work together in groups of **4**. The group is given a score, which they divide up, according to the amount of work each did, to get their individual scores.
Aoife, Brian, Conor and Dana worked together, and got a score of **10**. They decided it should be divided as:

Aoife: 1 Brian: 2 Conor: 3 Dana: 4.

They informed their lecturer of this, and he tried to enter these on Blackboard. But he is not very good with computers, and got ALL the scores wrong!
How many ways could this happen?

Eg we should have $\{a, b, c, d\} = \{1, 2, 3, 4\}$.
But $a=4$, $b=3$, $c=2$ & $d=1$ is a "derangement"
as is $a=2$, $b=1$, $c=4$, $d=3$.
But not $a=2$, $b=3$, $c=1$, $d=4$ because $d=4$ is correct.

What are all the possible derangements?

A **permutation** of a set is a re-ordering of it.
There are $n!$ permutations of a set with n elements.

A **derangement** is a permutation where no element is left in its original place.

Example: A B C.

Permutations: there are 6 in total $= 3! = 3 \times 2 \times 1$

$\underline{A}\ \underline{B}\ \underline{C}$
$\underline{A}$ C B
B A $\underline{C}$
B C A
C A B
$\left.\begin{matrix} \text{B C A} \\ \text{C A B} \end{matrix}\right\}$ these 2 are derangements
C $\underline{B}$ A

So we have found that $D_3 = 2$ by listing them all.

The study of **derangements** dates back to at least 1708. The old French card game called *rencontres* was a game of chance for two players, A and B:

- The players begin with a shuffled, full deck of 52 cards each.
- Each would take turns placing random cards on the table.
- If any of the cards matched, player A would win.
- If none of the cards matched, player B would win.

In 1708, Pierre de Montort (1678–1719) posed the problem: what is the probability that there would be no matches?

If we let D_{52} be the number of *derangements* of 52 cards then the solution is $D_{52}/52!$.

So B wins 36.7879412% of the time.

But 0.3678... is $1/e$ up to 18 digits

where e is Euler's Number.

Let D_n be the number of *derangements* of n objects. First we will work out formulae for D_1, D_2, D_3, and D_4.

$D_1 = 0$.

$D_2 = 1$

$D_3 = 2$ (a moment ago)

$D_4 = 4!\left(1 - 1/1! + 1/2! - 1/3! + 1/4!\right)$.

we will also see that

$D_4 = 9$. we can see this by listing them all (please do this!). We can also do this using PIE. (See OHP)

$$4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3}1! + \binom{4}{4}0!$$

$$= 4! - \frac{4!}{\cancel{3!}\,1!}\cancel{3!} + \frac{4!}{\cancel{2!}\,2!}\cancel{2!} - \frac{4!}{3!\,\cancel{1!}}\cancel{1!} + \frac{4!}{4!\,\cancel{0!}}\cancel{0!}$$

In general, the formula for D_n, the number of derangements of n objects is

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right).$$

This comes from the same reasoning as D_4 & application of the PIE.

Note that given n objects, the number of ways of permuting $n-k$, leaving k in place is

$$\binom{n}{k}(n-k)! = \frac{n!}{k!\,\cancel{(n-k)!}}\,\cancel{(n-k)!} = \frac{n!}{k!}$$

See also Exer 4 from Slide 20.

Suppose we have a set of n objects.

(a) How many k-permutations are there (with no repetition)? $P(n,k) = \frac{n!}{(n-k)!}$

(b) How many k-combinations are there (with no repetition)? $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

(c) How many k-permutations are there if repetition is allowed?

Ans: $n \times n \times n \times \cdots \times n = n^k$ (Recall bit strings).

(d) How many k-combinations are there if repetition is allowed?

Ans: $\binom{n+k-1}{k}$. why??

How many "words" can we make from the following sets of letters?

(i) {M,A,Y,O} $4 \times 3 \times 2 \times 1 = 4!$

(ii) {C,L,A,R,E} $5!$

(iii) {G,A,L,W,A,Y} 6 letters, but 2 the same : $6!/2$

(iv) {R,O,S,C,O,M,M,O,N}

AGLAWY same as AGLAWY.

(iv) we have 3 O's & two M's.

So need to be careful...

Let's consider the last example carefully: how many "words" can we make from letters in the set {C,M,M,N,O,O,O,R,S}?

If somehow the three O's were all distinguishable, and the two M's were distinguishable, the answer would be 9!.

But, since we can't distinguish identical letters,

- Let's choose which of the 9 positions we place the three O's. This can be done in $\binom{9}{3}$ ways.
- Now let's choose which of the remaining 6 positions we place the two M's. This can be done in $\binom{6}{2}$ ways.
- Now let's choose where to place the remaining 4 letters. This can be done in 4! ways.

By the Multiplicative Principle, the answer is

$$\binom{9}{3,2} = \binom{9}{3}\binom{6}{2}4! = \frac{9!}{3!\cancel{6!}}\,\frac{\cancel{6!}}{2!\cancel{4!}}\cancel{4!} = \frac{9!}{3!2!}$$

Multinomial coefficient

The number of different permutations of n objects, where there are n_1 indistinguishable objects of Type 1, n_2 indistinguishable objects of Type 2, ..., and n_k indistinguishable objects of Type k, is

$$\frac{n!}{(n_1!)(n_2!)\cdots(n_k!)}$$

Example (MA284 Semester 1 Examination, 2014/2015)

(i) Find the number of different arrangements of the letters in the place name WOLLONGONG.

(ii) How many of these arrangements start with the three O's;

(iii) How many contain the two G's consecutively;

(iv) How many *do not* contain the two G's consecutively?

(i) There is one W, three O's, two L's, 2 N's & 2 G's.

Ans: $\dfrac{10!}{3!\,2!\,2!\,2!\,1!} = 75{,}600$

Finished here, but the answers to (ii), (iii), (iv) added later.

Example (MA284 Semester 1 Examination, 2014/2015)

(i) Find the number of different arrangements of the letters in the place name WOLLONGONG.

(ii) How many of these arrangements start with the three O's;

(iii) How many contain the two G's consecutively;

(iv) How many *do not* contain the two G's consecutively?

(ii) Since each arrangement must start with three O's, we have just 7 left to arrange. There are two each of the L's, G's and N's, and one W. So the answer is:

$$\frac{7!}{2!2!2!1!} = 630 \text{ ways.}$$

Example (MA284 Semester 1 Examination, 2014/2015)

(i) Find the number of different arrangements of the letters in the place name WOLLONGONG.

(ii) How many of these arrangements start with the three O's;

(iii) How many contain the two G's consecutively;

(iv) How many *do not* contain the two G's consecutively?

(iii) To find the number of arrangements where the 'G's are together, just treat them as a single letter. Then we must arrange the 9 letters, where 'O' occurs 3 times, 'L' and 'N' twice, and W once. This can be done in

$$\frac{9!}{3!\,2!\,2!\,1!\,1!} = 15{,}120 \text{ ways.}$$

Example (MA284 Semester 1 Examination, 2014/2015)

(i) Find the number of different arrangements of the letters in the place name WOLLONGONG.

(ii) How many of these arrangements start with the three O's;

(iii) How many contain the two G's consecutively;

(iv) How many *do not* contain the two G's consecutively?

(iv) Use the answers to parts (i) and (iii) to get that it can be done in

$$\frac{10!}{3!2!2!2!} - \frac{9!}{3!2!2!} = 60,480 \text{ ways.}$$

Exercises

Most of these questions are based on exercises in Section 1.6 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. (a) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition of letters?"

(b) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition, if the letters must be in alphabetical order?"

Q2. How many integer solutions are there to the equation $x + y + z = 8$ for which

(a) x, y, and z are all positive?

(b) x, y, and z are all non-negative?

(c) x, y, and z are all greater than -3.

Q3. (Exercise 1.6.10 in the text-book) The Grinch sneaks into a room with 6 Christmas presents to 6 different people. He proceeds to switch the name-labels on the presents. How many ways could he do this if:

(a) No present is allowed to end up with its original label? Explain what each term in your answer represents.

(b) Exactly 2 presents keep their original labels? Explain.

(c) Exactly 5 presents keep their original labels? Explain.

Exercises

Q4. (MA284 Semester 1 Exam, 2016/2017) Let D_n be the number of derangements of n objects. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Q5. (MA284 Semester 1 Exam, 2015/2016) On Friday morning, before their Discrete Mathematics lecture, 8 students each leave one bag in the Cloakroom.
How many ways can their bags be returned to them?
How many ways can their bags be returned to them so that none of then gets their own bag back?
How many ways can their bags be returned to them so that exactly one of them gets their own bag back?

Q6. (MA284 Semester 1 Exam, 2015/2016) Give a formula for the number distinct permutations (arrangements) of all the letters in the word `BALLYGOBACKWARDS`.
How many of these begin with an "L"?
How many have all the vowels together?
How many have all the letters in alphabetical order?