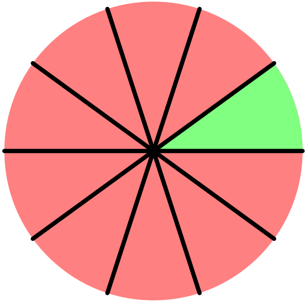
MA284 : Discrete Mathematics

Week 6: Advance applications of the PIE

http://www.maths.nuigalway.ie/~niall/MA284

11 and 13 of October, 2017



See also §1.6 of Levin's *Discrete Mathematics: an open introduction*.

Assignment 1 is now closed

Your grades are available from the Blackboard Grade Centre.
The average score was ...

. . . .

Assignment 2 is now open!

To access the assignment, go to
`http://mathswork.nuigalway.ie/webwork2/1718-MA284`

Your USERNAME and PASSWORD are:
There are **20** questions. You may attempt each one up to **10** times.
This assignment contributes 10% to your final grade for Discrete Mathematics.

Deadline: 5pm, Friday, 27 October.

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Please complete the online survey on MA284 to provide us with feedback on improving it. The link is in an announcement on Blackboard.

Last week we had the following question

How many ways can you share **n** apples among your **k** lecturers?

- This is the same as finding the number of ways we can arrange n apples (stars), divided into k groups, separated by $k-1$ bars.
- Any way can be written with $n+k-i$ symbols (n stars and $k-1$ bars): we just have to choose where to put the $k-1$ bars. This can be done in $\binom{n+k-1}{k-1} = \binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(k-1)!}$ ways.
- This can also be thought of as a *multiset*: a set of objects, where each object can appear more than once.
- And it can be framed as the number of solutions to the **non-negative integer problem:**

$$x_1 + x_2 + \cdots + x_k = n.$$

These three problems have the same number of solutions

The number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + \cdots + x_k + x_{k+1} = n. \tag{1}$$

is the same as the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + \cdots + x_k \leq n, \tag{2}$$

which is the same as the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + \cdots + x_k < n + 1, \tag{3}$$

We ended with solving variants on these problems that have constraints, for example

(a) where each of the x_k is at least 2;

(b) where each of the x_k is at most 5.

Solving the latter brought us back to the PIE.

Recall that

- $|X|$ denotes the number of elements in the set X.
- $X \cup Y$ (the *union of X and Y*) is the set of all elements that belong to **either** X or Y.
- $A \cap B$ (the *intersection of X and Y*) is the set of all elements that belong to *both* X and Y.

The **Principle of Inclusion/Exclusion (PIE)** for two sets, A and B, is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For for three sets, A, B and C, the PIE is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example:

We will now use this idea to solve non-negative integer problems, with *upper bounds* on the solution.

Example

Students work together in groups of 3 on a Discrete Mathematics assignment. The group is given a score, which they divide up (according to the amount of work each did) to get their individual scores. The maximum individual score is 4 marks.

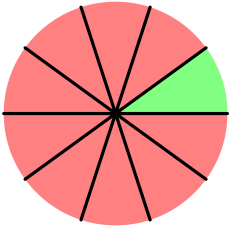
Aoife, Brian and Conor worked together as a group. Their group got a score of $\mathbf{11}$. How many ways can this be divided out?

The PIE works for larger numbers of sets too, although it gets a little messy to write down. For **4** sets, we can think of it as

$$
\begin{aligned}
|A \cup B \cup C \cup D| = &\text{(the sum of the sizes of each single set)} \\
&- \text{(the sum of the sizes of each } \textbf{intersection} \text{ of 2 sets)} \\
&+ \text{(the sum of the sizes of each } \textbf{intersection} \text{ of 3 sets)} \\
&- \text{(the sum of the sizes of } \textbf{intersection} \text{ of all 4 sets)}
\end{aligned}
$$

Example (Example 1.6.2 of text-book)

How many ways can we distribute 10 slices of pie(!) to 4 kids so that no kid gets more than 2 slices?



Not all such problems have easy solution solutions.

Example

How many non-negative integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \qquad \text{if...}$$

1. There are no restrictions (other than each x_i being an *nni*).
2. $0 \leq x_i \leq 3$ for each i.

For **Assignment 2** of MA840 (*Indiscreet Mathematics*), students work together in groups of **4**. The group is given a score, which they divide up, according to the amount of work each did, to get their individual scores.
Aoife, Brian, Conor and Dana worked together, and got a score of **10**. They decided it should be divided as:

Aoife: 1   Brian: 2   Conor: 3   Dana: 4.

They informed their lecturer of this, and he tried to enter these on Blackboard. But he is not very good with computers, and got ALL the scores wrong!
How many ways could this happen?

A **permutation** of a set is a re-ordering of it.
There are $n!$ permutations of a set with n elements.

A **derangement** is a permutation where no element is left in its original place.

Example:

The study of **derangements** dates back to at least 1708. The old French card game called *rencontres* was a game of chance for two players, A and B:

- The players begin with a shuffled, full deck of 52 cards each.
- Each would take turns placing random cards on the table.
- If any of the cards matched, player A would win.
- If none of the cards matched, player B would win.

In 1708, Pierre de Montort (1678–1719) posed the problem: what is the probability that there would be no matches?

If we let D_{52} be the number of *derangements* of 52 cards then the solution is $D_{52}/52!$.

Let D_n be the number of *derangements* of n objects. First we will work out formulae for D_1, D_2, D_3, and D_4.

In general, the formula for D_n, the number of derangements of n objects is

$$D_n = n!\big(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\big).$$

Suppose we have a set of n objects.

(a) How many k-permutations are there (with no repetition)?

(b) How many k-combinations are there (with no repetition)?

(c) How many k-permutations are there if repetition is allowed?

(d) How many k-combinations are there if repetition is allowed?

How many "words" can we make from the following sets of letters?

(i) $\{\mathtt{M},\mathtt{A},\mathtt{Y},\mathtt{O}\}$

(ii) $\{\mathtt{C},\mathtt{L},\mathtt{A},\mathtt{R},\mathtt{E}\}$

(iii) $\{\mathtt{G},\mathtt{A},\mathtt{L},\mathtt{W},\mathtt{A},\mathtt{Y}\}$

(iv) $\{\mathtt{R},\mathtt{O},\mathtt{S},\mathtt{C},\mathtt{O},\mathtt{M},\mathtt{M},\mathtt{O},\mathtt{N}\}$

Let's consider the last example carefully: how many "words" can we make from letters in the set $\{\texttt{C},\texttt{M},\texttt{M},\texttt{N},\texttt{O},\texttt{O},\texttt{O},\texttt{R},\texttt{S}\}$?

If somehow the three O's were all distinguishable, and the two M's were distinguishable, the answer would be $9!$.

But, since we can't distinguish identical letters,

- Let's choose which of the 9 positions we place the three O's. This can be done in $\binom{9}{3}$ ways.
- Now let's choose which of the remaining 6 positions we place the two M's. This can be done in $\binom{6}{2}$ ways.
- Now let's choose where to place the remaining 4 letters. This can be done in $4!$ ways.

By the Multiplicative Principle, the answer is

$$\binom{9}{3}\binom{6}{2}4! = \frac{9!}{3!6!}\frac{6!}{2!4!}4! = \frac{9!}{3!2!}$$

Multinomial coefficient

The number of different permutations of n objects, where there are n_1 indistinguishable objects of Type 1, n_2 indistinguishable objects of Type 2, ..., and n_k indistinguishable objects of Type k, is

$$\frac{n!}{(n_1!)(n_2!)\cdots(n_k!)}$$

Example (MA284 Semester 1 Examination, 2014/2015)

(i) Find the number of different arrangements of the letters in the place name `WOLLONGONG`.

(ii) How many of these arrangements start with the three `O`'s;

(iii) How many contain the two `G`'s consecutively;

(iv) How many *do not* contain the two `G`'s consecutively?

Most of these questions are based on exercises in Section 1.6 of Levin's *Discrete Mathematics*. Solutions are also available from that book.

Q1. (a) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition of letters?"
(b) How many 6-letter words can you make using some or all of the 5 letters `MATHS`, allowing repetition, if the letters must be in alphabetical order?"

Q2. How many integer solutions are there to the equation $x + y + z = 8$ for which

(a) x, y, and z are all positive?
(b) x, y, and z are all non-negative?
(c) x, y, and z are all greater than -3.

Q3. (Exercise 1.6.10 in the text-book) The Grinch sneaks into a room with 6 Christmas presents to 6 different people. He proceeds to switch the name-labels on the presents. How many ways could he do this if:

(a) No present is allowed to end up with its original label? Explain what each term in your answer represents.
(b) Exactly 2 presents keep their original labels? Explain.
(c) Exactly 5 presents keep their original labels? Explain.

Q4. (MA284 Semester 1 Exam, 2016/2017) Let D_n be the number of derangements of n objects. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Q5. (MA284 Semester 1 Exam, 2015/2016) On Friday morning, before their Discrete Mathematics lecture, 8 students each leave one bag in the Cloakroom.
How many ways can their bags be returned to them?
How many ways can their bags be returned to them so that none of then gets their own bag back?
How many ways can their bags be returned to them so that exactly one of them gets their own bag back?

Q6. (MA284 Semester 1 Exam, 2015/2016) Give a formula for the number distinct permutations (arrangements) of all the letters in the word `BALLYGOBACKWARDS`.
How many of these begin with an "L"?
How many have all the vowels together?
How many have all the letters in alphabetical order?