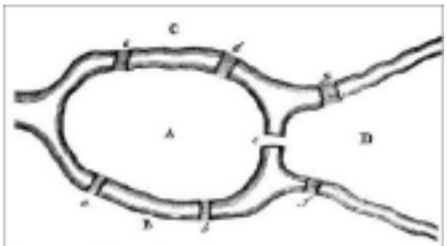
MA284 : Discrete Mathematics

Week 7: Introduction to Graph Theory.

http://www.maths.nuigalway.ie/~niall/MA284/

18 and 20 October, 2017



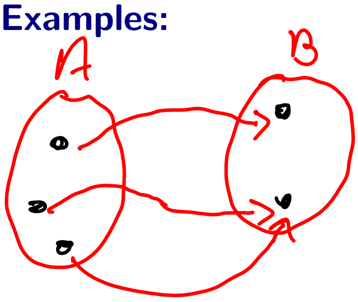
The seven bridges of Königsberg

See also §1.6 and §4.0 of Levin's *Discrete Mathematics: an open introduction*.

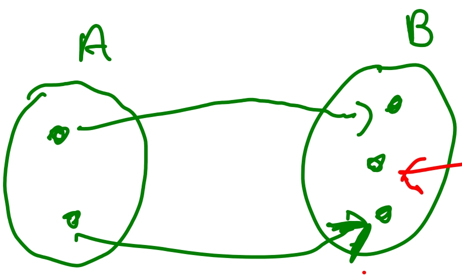
Finished here Wednesday.

When every element of B is the image of some element of A, we say that the function is *SURJECTIVE* (also called "onto").

Examples:



Surjective.



not the image of anything : So f is not surjective

When every element of B is the image of some element of A, we say that the function is *SURJECTIVE* (also called "onto").

Examples:

$A = \{-2, -1, 0, 1, 2\}$

$B = \{0, 1, 2, 3, 4\}$

$f(x) = x + 2$ is surjective

But $f(x) = |x|$ is not.

(There is no x in A for which $|x| = 4$.)

When no two elements of A have the same image in B, we say that the function is *INJECTIVE* (also called “one-to-one”).

Examples:

$$A = \{-2, -1, 0, 1, 2\}$$

$$B = \{0, 1, 2, 3, 4\}$$

$f(x) = x + 2$ is one-to-one (Injective)

but

$f(x) = |x|$ is not, since, eg $|-2| = |2|$.

Bijection

The function $f : A \to B$ is a **BIJECTION** if it is both *surjective* and *injective*. Then f defines a *one-to-one correspondence* between A and B.

From the previous eg,

$f(x) = x + 2$ is a Bijection

Eg, the function that maps everyone here to their student ID is a Bijection

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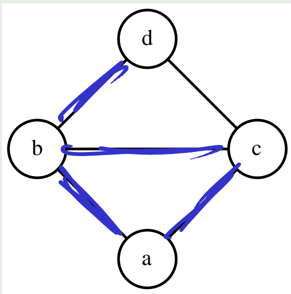
Given two sets, A & B, such that there is a bijection from A to B, we say they are in BIJECTIVE CORRESPONDENCE.

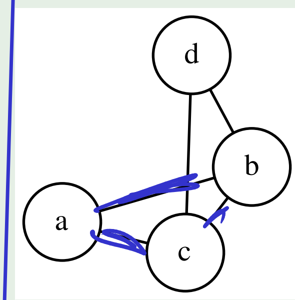
Two graphs are *EQUAL* if the have exactly the same Edge and Vertex sets. That is *it is not important how we draw them*, how where we position the vertices, the length of the edges, etc.

Example (Section 4.1 of text-book)

Show that the two graphs given below are *equal*

$G_1 = (V_1, E_1)$





$G_2 = (V_2, E_2)$

G_1 $V_1 = \{a, b, c, d\}$

$E_1 = \{\{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}\}$

G_2 $V_2 = \{a, b, c, d\}$

$E_2 = \{\{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}\}$

Isomorphism

An *ISOMORPHISM* between two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is a *bijection* $f : V_1 \to V_2$ between the vertices in the graph such that, if $\{a, b\}$ is an edge in G_1, then $\{f(a), f(b)\}$ is an edge in G_2.
Two graphs are *ISOMORPHIC* if there is an isomorphism between them. In that case, we write $G_1 \cong G_2$.

Example (Example 4.1.1 of text-book)

Show that the graphs

$$G_1 = \{V_1, E_1\}, \text{ where } V_1 = \{a, b, c\} \text{ and } E_1 = \{\{a, b\}, \{a, c\}, \{b, c\}\};$$

$$G_2 = \{V_2, E_2\} \text{ where } V_2 = \{u, v, w\}, \text{ and } E_2 = \{\{u, v\}, \{u, w\}, \{v, w\}\}$$

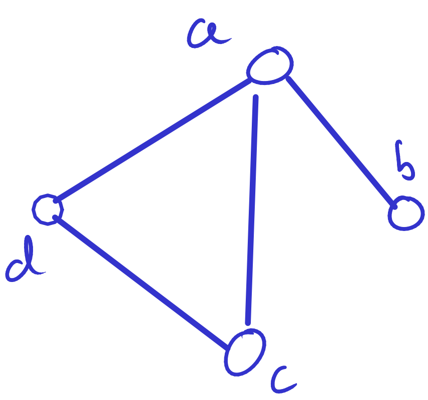
are not *equal* but are *isomorphic*.

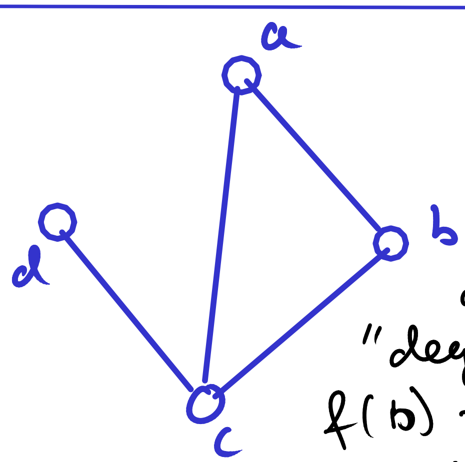
They are not EQUAL since $\{a, b, c\} \neq \{u, v, w\}$
They are isomorphic, with $f(a) = u$, $f(b) = v$
$f(c) = w$.

Example (Example 4.1.3 from text-book)

Decide whether the graphs $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are equal or isomorphic, where
$V_1 = \{a, b, c, d\}$, $E_1 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}\}$ and
$V_2 = \{a, b, c, d\}$, $E_2 = \{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}$

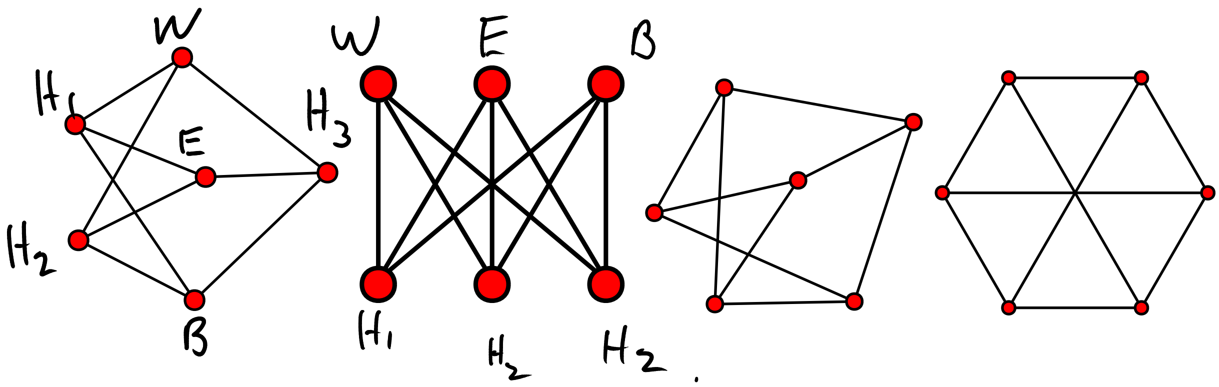
They are not equal. Eg, $\{a,d\} \in E_1$ but $\{a,d\} \notin E_2$.





These are isomorphic. Since both b in G_1 & d in G_2 have "degree 1", let $f(b) = d$, Then set $f(a) = c$, and (eg) $f(c) = b$, $f(d) = a$.

When we give a graph without labeling the vertices, we are really talking about *all* graphs that are **isomorphic** to the one we have just drawn. For example, when we draw the following graph, we mean it to represent all those graphs that are isomorphic to the *Water-Electricity-Broadband* graph.

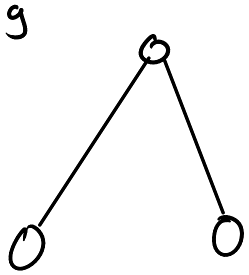


Other than the Königsberg Bridges example, all the graphs we have looked at so far

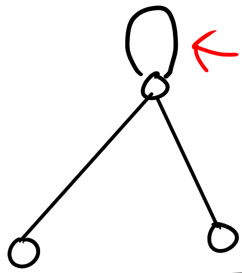
1. have no *loops* (i.e., no edge from a vertex to itself).
2. have no repeated edges (i.e., there is at most one edge between each pair of vertices).

Such graphs are called *SIMPLE* graphs. But because they are the most common, unless we say otherwise, when we say "graph" we mean "simple graph".

Eg

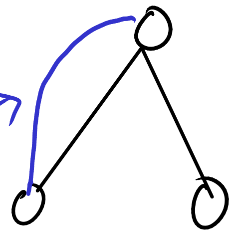


This is simple

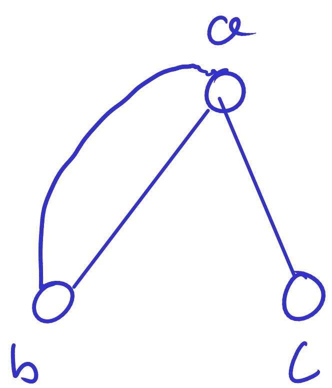


loop, so not a simple graph

Repeated edge.



If a graph does have repeated edges, like in the Königsberg example, we call it a *MULTIGRAPH*. Then the list of edges is not a set, since some elements are repeated: it is a multiset (see Week 5).



$V = \{a, b, c\}$

$E = \{ \underbrace{\{a,b\}, \{a,b\}, \{a,c\}}_{} \}$

multiset.

Finished here Friday.

Q1. (Exercise 4.1.1 from text-book) If 10 people each shake hands with each other, how many handshakes took place? What does this question have to do with graph theory?

Q2. (Exercise 4.0.2 of text-book *and* MA284/MA204 Semester 1 Exam, 2015/2016) Among a group of five people, is it possible for everyone to be friends with exactly two of the other people in the group?
Is it possible for everyone to be friends with exactly three of the other people in the group?
Explain your answers carefully.

Q3. Is it possible for two *different* (non-isomorphic) graphs to have the same number of vertices and the same number of edges? What if the degrees of the vertices in the two graphs are the same (so both graphs have vertices with degrees 1, 2, 2, 3, and 4, for example)? Draw two such graphs or explain why not.

Q4. Are the two graphs below equal? Are they isomorphic? If they are isomorphic, give the isomorphism. If not, explain.

Graph 1: $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, e\}\}$.

Graph 2: