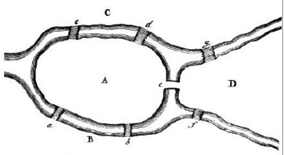
MA284 : Discrete Mathematics

Week 7: Introduction to Graph Theory.

http://www.maths.nuigalway.ie/~niall/MA284/

18 and 20 October, 2017



The seven bridges of Königsberg

See also §1.6 and §4.0 of Levin's *Discrete Mathematics: an open introduction*.

ASSIGNMENT 2 IS OPEN

To access the assignment, go to
http://mathswork.nuigalway.ie/webwork2/1718-MA284

There are **20** questions. You may attempt each one up to **10** times.
This assignment contributes 10% to your final grade for Discrete Mathematics.

Deadline: 5pm, Friday, 27 October.

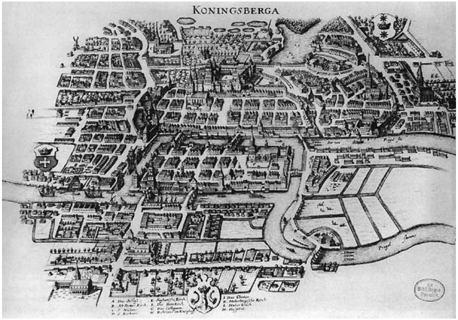
. . . .

Please complete the online survey on MA284 to provide us with feedback on improving it. The link is in an announcement on Blackboard.

Graph Theory is a branch of mathematics that is several hundred years old. Many of its discoveries were motivated by practical problems, such as determining the smallest number of colours needed to colour a map.

It is unusual in that its beginnings can be traced to a precise date.

Königsberg in Prussia (now Kaliningrad, Russia) had seven bridges. Is it possible it walk through the town in such a way that you cross each bridge once and only once?



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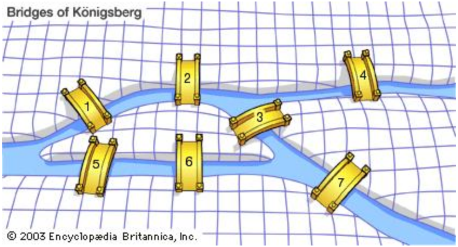

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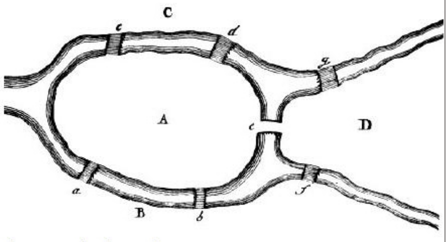

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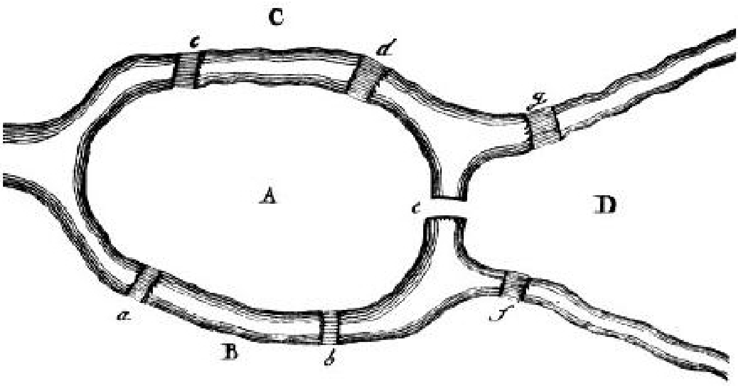
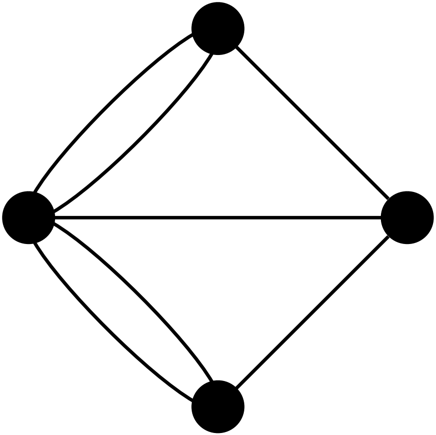


The seven bridges of Königsberg

Here is another way of stating the same problem. Consider the following picture, which shows 4 dots connected by some lines.



Is it possible to trace over each line once and only once (without lifting up your pencil)? You must start and end on one of the dots.

Graph

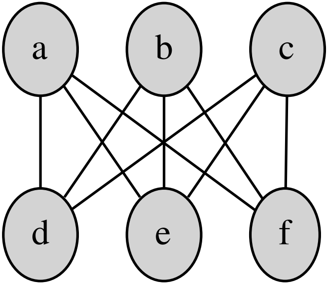
A *GRAPH* is a collection of

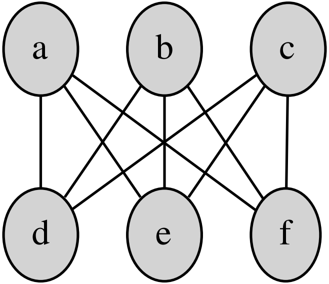
- "vertices" (or "nodes"), which are the "dots" in the above diagram.
- "edges" joining pair of vertices.

If the graph is called G (say), we often define it in terms of its *edge set* and *vertex set*. That is we write

$$G = (V, E),$$

where V is the set of vertices and E is the set of edges.

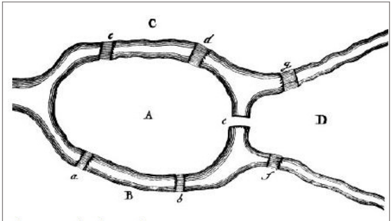




If two vertices are connected by an edge, we say they are *adjacent*.

Graphs are used to represent collections of objects where there is a special relationship between certain pairs of objects.

For example, in the Königsberg problem, the land-masses are vertices, and the edges are bridges.



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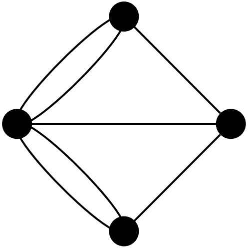


(Example 4.0.1 of the text-book)

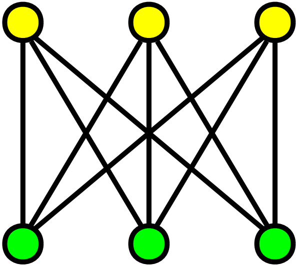
Aoife, Brian, Conor, David and Edel are students in a *Indiscrete Mathematics* module.

- Aoife and Conor worked together on their assignment.
- Brian and David also worked together on their assignment.
- Edel helped everyone with their assignments.

Represent this situation with a graph.

The Three Utilities Problem; also Example 4.0.2 in the text-book.

We must make water, power and broadband connections to three houses.
Is it possible to do this without the conduits crossing?



Sketch the graph with Vertex Set $V = \{a, b, c, d\}$ and Edge Set

$$E = \big\{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\big\}$$

Recall the $f : A \to B$ is a *function* that maps every elements of the set A onto some element of set B. (We call A the "domain", and B the "codomain".)
Each element of A gets mapped to exactly one element of B.

If $f(a) = b$ where $a \in A$ and $b \in B$, we say that "the image of a is b".
Or, equivalently, "b is the image of a".

Examples:

When every element of B is the image of some element of A, we say that the function is *SURJECTIVE* (also called "onto").

Examples:

When no two elements of A have the same image in B, we say that the function is *INJECTIVE* (also called "one-to-one").

Examples:

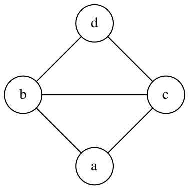
Bijection

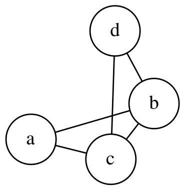
The function $f : A \to B$ is a **BIJECTION** if it is both *surjective* and *injective*. Then f defines a *one-to-one correspondence* between A and B.

Two graphs are *EQUAL* if the have exactly the same Edge and Vertex sets. That is *it is not important how we draw them*, how where we position the vertices, the length of the edges, etc.

Example (Section 4.1 of text-book)

Show that the two graphs given below are *equal*





Isomorphism

An *ISOMORPHISM* between two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is a *bijection* $f : V_1 \to V_2$ between the vertices in the graph such that, if $\{a, b\}$ is an edge in G_1, then $\{f(a), f(b)\}$ is an edge in G_2.
Two graphs are *ISOMORPHIC* if there is an isomorphism between them. In that case, we write $G_1 \cong G_2$.

Example (Example 4.1.1 of text-book)

Show that the graphs

$$G_1 = \{V_1, E_1\}, \text{ where } V_1 = \{a, b, c\} \text{ and } E_1 = \{\{a, b\}, \{a, c\}, \{b, c\}\};$$

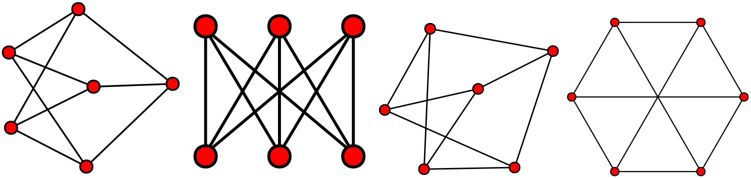
$$G_2 = \{V_2, E_2\} \text{ where } V_2 = \{u, v, w\}, \text{ and } E_2 = \{\{u, v\}, \{u, w\}, \{v, w\}\}$$

are not *equal* but are *isomorphic*.

Example (Example 4.1.3 from text-book)

Decide whether the graphs $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are equal or isomorphic, where
$V_1 = \{a, b, c, d\}$, $E_1 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}\}$ and
$V_2 = \{a, b, c, d\}$, $E_2 = \{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}$

When we give a graph without labeling the vertices, we are really talking about *all* graphs that are **isomorphic** to the one we have just drawn. For example, when we draw the following graph, we mean it to represent all those graphs that are isomorphic to the *Water-Electricity-Broadband* graph.



Other than the Königsberg Bridges example, all the graphs we have looked at so far

1. have no *loops* (i.e., no edge from a vertex to itself).
2. have no repeated edges (i.e., there is at most one edge between each pair of vertices).

Such graphs are called *SIMPLE* graphs. But because they are the most common, unless we say otherwise, when we say "graph" we mean "simple graph".

If a graph does have repeated edges, like in the Königsberg example, we call it a *MULTIGRAPH*. Then the list of edges is not a set, since some elements are repeated: it is a multiset (see Week 5).

Q1. (Exercise 4.1.1 from text-book) If 10 people each shake hands with each other, how many handshakes took place? What does this question have to do with graph theory?

Q2. (Exercise 4.0.2 of text-book *and* MA284/MA204 Semester 1 Exam, 2015/2016) Among a group of five people, is it possible for everyone to be friends with exactly two of the other people in the group?
Is it possible for everyone to be friends with exactly three of the other people in the group?
Explain your answers carefully.

Q3. Is it possible for two *different* (non-isomorphic) graphs to have the same number of vertices and the same number of edges? What if the degrees of the vertices in the two graphs are the same (so both graphs have vertices with degrees 1, 2, 2, 3, and 4, for example)? Draw two such graphs or explain why not.

Q4. Are the two graphs below equal? Are they isomorphic? If they are isomorphic, give the isomorphism. If not, explain.

Graph 1: $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, e\}\}$.

Graph 2: