Week 10: Colouring graphs, and Euler’s paths

13 and 15 November 2019

1 Colouring
   - The Four Colour Theorem

2 Graph colouring
   - Chromatic Number

3 Algorithms
   - Greedy algorithm
   - Welsh-Powell Algorithm

4 Eulerian Paths and Circuits

5 Next week: Hamiltonian Paths and Cycles

6 Exercises

See also Chapter 4 of Levin’s *Discrete Mathematics: an open introduction*. 
ASSIGNMENTS 4 and 5 are open

Access them at
http://mathswork.nuigalway.ie/webwork2/1920-MA284

You have 10 attempts for each. Not all questions carry the same score.

Both assignments contribute 8% to your final grade for Discrete Mathematics.

Deadlines:

Assignment 4: 5pm, Friday 15 November.
Assignment 5: 5pm, Friday 29 November.

For more information, see Blackboard, or
http://www.maths.nuigalway.ie/~niall/MA284
[From textbook, p184]. Here is a map of the (fictional) country “Euleria”. Colour it so that adjacent regions are coloured differently. What is the fewest colours required?
There are maps that can be coloured with

- A single colour:
- Two colours (e.g., the island of Ireland):
- Three colours:
- Four colours:

*Just about every other map.*
It turns out that there is no map that needs more than 4 colours. This is the famous **Four Colour Theorem**, which was originally conjectured by the British/South African mathematician and botanist, Francis Guthrie who at the time was a student at University College London.

He told one of his mathematics lecturers, Augustus de Morgan, who, on **23 October, 1852**, wrote to friend William Rowan Hamilton, who was in Dublin:
From https://en.wikipedia.org/wiki/Four_color_theorem

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A student of mine asked me to day to give him a reason for a fact which I did not know was a fact and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured-four colours may be wanted, but not more-the following is his case in which four are wanted.
Query: cannot a necessity for five or more be invented... What do you say? And has it, if true been noticed?
My pupil says he guessed it in colouring a map of England... The more I think of it, the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did...

De Morgan needn’t have worried: a proof was not produced until 1976. It is very complicated, and relies heavily on computer power.
To get a sense of *why* it might be true, try to draw a map that needs 5 colours.

Our interest is not in trying to prove the Four Colour Theorem, but in how it is related to Graph Theory.
If we think of a map as a way of showing which regions share borders, then we can represent it as a *graph*, where

- A vertex in the graph corresponds to a region in the map;
- There is an edge between two vertices in the graph if the corresponding regions share a border.

**Example:**

![Graph colouring example](image-url)
Colouring regions of a map corresponds to colouring vertices of the graph. Since neighbouring regions in the map must have different colours, so too adjacent vertices in the graph must have different colours.

More precisely

**Vertex Colouring:** An assignment of colours to the vertices of a graph is called a *VERTEX COLOURING*.

**Proper Colouring:** If the vertex colouring has the property that adjacent vertices are coloured differently, then the colouring is called *PROPER*.

Lots different proper colourings are possible. If the graph as $v$ vertices, the clearly at most $v$ colours are needed. However, usually, we need far fewer.
Graph colouring

Examples:

- Colouring, but not proper
- Proper colouring, but not minimal

Proper and minimal
The smallest number of colours needed to get a proper vertex colouring if a graph $G$ is called the CHROMATIC NUMBER of the graph, written $\chi(G)$.

**Example:** Determine the Chromatic Number of the graphs $C_2$, $C_3$, $C_4$ and $C_5$.

- $\chi(C_2) = 2$

- $\chi(C_3) = 3$

- $\chi(C_4) = 2$

- $\chi(C_5) = 3$
Example: Determine the Chromatic Number of the $K_n$ and $K_{p,q}$ for any $n, p, q$.

$K_n$ is the complete graph on $n$ vertices. Every vertex is a neighbour of every other one. So $\chi(K_n) = n$ for all $n$.

$K_{p,q}$ is the complete bipartite graph. $\chi(K_{p,q}) = 2$ for any $p, q$.

$K_{3,3}$
In general, calculating $\chi(G)$ is not easy. There are some ideas that can help. For example, it is clearly true that, if a graph has $v$ vertices, then

$$1 \leq \chi(G) \leq v.$$

If the graph happens to be complete, then $\chi(G) = v$. If it is not complete the we can look at cliques in the graph.

**Clique:** A *CLIQUE* is a subgraph of a graph all of whose vertices are connected to each other.

Examples of cliques include $\{b, c, d\}$ and $\{a, b, d, f\}$.

So $\chi(G)$ is at least 4.
The **CLIQUE NUMBER** of a graph, $G$, is the number of vertices in the largest clique in $G$.

From the last example, we can deduce that

**LOWER BOUND**: The chromatic number of a graph is at least its clique number.

We can also get a useful upper bound. Let $\Delta(G)$ denote the largest degree of any vertex in the graph, $G$,

**UPPER BOUND**: $\chi(G) \leq \Delta(G) + 1$.

*Why?* This is called **Brooks’ Theorem**, and is Thm 4.5.5 in the text-book:

http://discrete.openmathbooks.org/dmoi3/sec_coloring.html
In general, finding a proper colouring of a graph is *hard*. There are some algorithms that are efficient, but not optimal. We’ll look at two:

1. The *Greedy algorithm*.
2. The *Welsh-Powell algorithm*.

The **Greedy algorithm** is simple and efficient, but the result can depend on the ordering of the vertices. The **Welsh-Powell** is slightly more complicated, but can give better colourings.
The GREEDY ALGORITHM

1. Number all the vertices. Number your colours.
2. Give a colour to vertex 1.
3. Take the remaining vertices in order. Assign each one the lowest numbered colour, that is different from the colours of its neighbours.

Example: Apply the GREEDY ALGORITHM to colouring the following graph (the cubical graph, $Q_3$):

\[X(Q_3) = 2.\]
Welsh-Powell Algorithm

1. List all vertices in decreasing order of their degree (so largest degree is first). If two or more have the same degree, list those any way.
2. Colour to the first listed vertex (with an as-yet unused colour).
3. Work down the list, giving that colour to all vertices \textit{not} connected to one previously coloured.
4. Cross coloured vertices off the list, and return to the start of the list.

Example: Colour this graph using both GREEDY and WELSH-POWELL:

\begin{itemize}
  \item \textbf{GREEDY:}
    \begin{itemize}
      \item Colour 1 with colour 1.
      \item Colour 2 with colour 2.
      \item Colour 3 with colour 3.
      \item Colour 4 with colour 4.
      \item Colour 5 with colour 5.
    \end{itemize}
  \item \textbf{WELSH-POWELL:}
    \begin{itemize}
      \item Colour 1 with colour 1.
      \item Colour 2 with colour 2.
      \item Colour 3 with colour 3.
      \item Colour 4 with colour 4.
      \item Colour 5 with colour 5.
    \end{itemize}
\end{itemize}