§1 Interpolation

# §1.2 Finding the interpolant

MA378/531 – Numerical Analysis II ("NA2")

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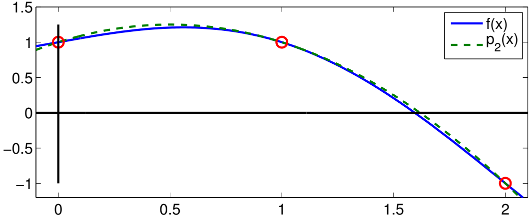
Joseph-Louis Lagrange, born 1736 in Turin, died 1813 in Paris. He made great contributions to many areas of Mathematics, including *Calculus of Variations*.

**Example**

Show that the polynomial of degree 2 that interpolates $f(x) = 1 - x + \sin(\pi x/2)$ at the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ is $p_2 = -x^2 + x + 1$.



It is not hard to convince ourselves that $-x^2 + x + 1.$ is the solution to the above PIP. But how to we know we have found the *only* solution? More generally, *under what conditions is there exactly one polynomial that solves the PIP*?

As a first step, we'll prove the following:

**Lemma**

*If $p_n \in \mathcal{P}_n$ has $n + 1$ zeros, then $p_n \equiv 0$ (i.e., $p_n(x) = 0$ for all $x$).*

**Theorem (There is a unique solution to the PIP)**

*There is at most one polynomial of degree* $\leq n$ *that interpolates the* $n+1$ *points* $(x_0, y_0)$, $(x_1, y_1)$, *...*, $(x_n, y_n)$ *where* $x_0$, $x_1$, *...*, $x_n$ *are distinct.*

Now we want to solve the PIP. It turns out that the most obvious approach may not be the best.

Suppose we are trying to solve the problem as follows: *find $p_2$ such that*

$$p_2(x_0) = y_0, \quad p_2(x_1) = y_1, \quad \textit{and} \quad p_2(x_2) = y_2.$$

Since $p_2(x)$ is of the form $a_0 + a_1 x + a_2 x^2$, this just amounts the finding the values of the coefficients $a_0$, $a_1$, and $a_2$. One might be tempted to solve for them using the system of equations

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 &= y_2 \end{aligned}$$

This is known as the *Vandermonde System*.

Writing

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 &= y_2 \end{aligned}$$

in matrix-vector format we get

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \quad \text{or} \quad V\boldsymbol{a} = \boldsymbol{y}. \tag{1}$$

But this may not be a good idea. (*Unfortunately, to see exactly why, you needed to have studied MA385. If you didn't, you can skip the next bit*).

In MA385 we learned about the relationship between the *condition number* of a matrix, $V$, and the relative error in the (numerical) solution to a matrix-vector equation with $V$ as the coefficient matrix. The condition number is $\kappa(V) = \|V\|\|V^{-1}\|$, for some subordinate matrix norm $\|\cdot\|$.

**Example (Stewart's "Afternotes...", Lecture 19)**

Suppose $x_0 = 100$, $x_1 = 101$ and $x_2 = 102$. Then it is not hard to check that

$$\|X\|_\infty = \max_i \sum_j |X_{ij}| = 10,507.$$

Also,

$$V^{-1} = \frac{1}{2}\begin{pmatrix} 10302 & -20400 & 10100 \\ -203 & 404 & -201 \\ 1 & -2 & 1 \end{pmatrix},$$

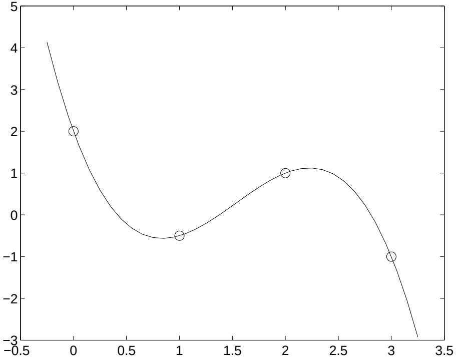
so $\|V^{-1}\|_\infty = 20401$. So $\kappa(V) = 214,353,307$.

We'll now look at a much easier method for solving the Polynomial Interpolation Problem. As a by-product, we get a constructive proof of the existence of a solution to the PIP. (Here "constructive" means that we'll prove it exists by actually computing it).

## Example

Consider the problem: *find* $p_3 \in \mathcal{P}_3$ *such that*

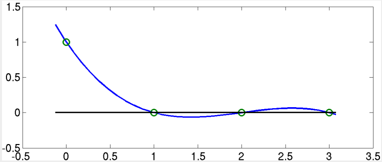
$$p_3(0) = 2, \quad p_3(1) = -1/2, \quad p_3(2) = 1, \quad p_3(3) = -1.$$



Here is an easier problem to solve.

Find $L_0 \in \mathcal{P}_3$ such that

$$L_0(0) = 1, \quad L_0(1) = 0, \quad L_0(2) = 0, \quad L_0(3) = 0.$$

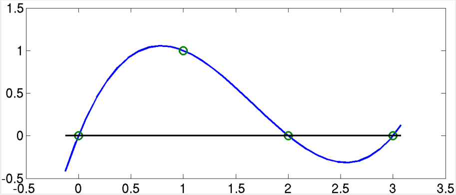



Because $L_0$ is a cubic and has zeros at $x = 1, 2, 3$ it is of the form $L_0(x) = C(x-1)(x-2)(x-3)$. Choosing $C$ so that $L_0(0) = 1$, we get

$L_0(x) =$

Similarly, let $L_1 \in \mathcal{P}_3$ be the cubic polynomial such that

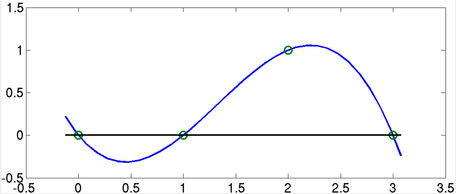
$$L_1(0) = 0, \quad L_1(1) = 1, \quad L_1(2) = 0, \quad L_1(3) = 0,$$




Then

$$L_1(x) =$$

In the same style, let $L_2(x_i) = \begin{cases} 1 & i = 2 \\ 0 & i = 0, 1, 3 \end{cases}$



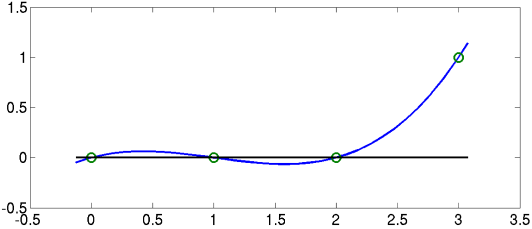

$$L_2(x) =$$

Finally, if we define

$$L_3(x_i) = \begin{cases} 1 & i = 3 \\ 0 & i = 0, 1, 2 \end{cases},$$

then clearly,

$$L_3(x) = \frac{(x-0)(x-1)(x-3)}{(3-0)(3-1)(3-2)} = \prod_{j=0, j\neq 3}^{n} \frac{(x - x_j)}{(x_3 - x_j)}.$$



Because each of $L_0$, $L_1$, $L_2$, and $L_3$ is a cubic, so too is any linear combination of them. So

$$p_3(x) = 2L_0(x) - (1/2)L_1(x) + (1)L_2(x) + (-1)L_3(x),$$

is a cubic. Furthermore

$$\begin{array}{rcllll}
p_3(0) & = & 2L_0(0) & -(1/2)L_1(0) & +(1)L_2(0) & +(-1)L_3(0) \\
& = & 2(1) & -(1/2)(0) & +(1)(0) & +(-1)(0) \\
& = & 2, & & & \\
p_3(1) & = & 2L_0(1) & -(1/2)L_1(1) & +(1)L_2(1) & +(-1)L_3(1) \\
& = & 2(0) & -(1/2)(1) & +(1)(0) & +(-1)(0) \\
& = & -1/2, & & & \\
p_3(2) & = & 2L_0(2) & -(1/2)L_1(2) & +(1)L_2(2) & +(-1)L_3(2) \\
& = & 2(0) & -(1/2)(0) & +(1)(1) & +(-1)(0) \\
& = & 1, & & & \\
p_3(3) & = & 2L_0(3) & -(1/2)L_1(3) & +(1)L_2(3) & +(-1)L_3(3) \\
& = & 2(0) & -(1/2)(0) & +(1)(0) & +(-1)(1) \\
& = & -1. & & &
\end{array}$$

Thus $p_3$ solves the problem!

We can generalise this idea to solve any PIP using what is called *Lagrange* interpolation. We'll now look how to solve the general problem.

**Definition (Lagrange Polynomials)**

The **Lagrange Polynomials** associated with $x_0 < x_1 < \cdots < x_n$ is the set $\{L_i\}_{i=0}^{n}$ of polynomials in $\mathcal{P}_n$ such that

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{2a}$$

and are given by the formula

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}. \tag{2b}$$

## Definition

The **Lagrange form of the Interpolating Polynomial**

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x), \tag{3a}$$

or

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x). \tag{3b}$$

*Take care not to confuse the Lagrange Polynomials, which are the $L_i$ with the Lagrange Interpolating Polynomial, which is the $p_n$ defined in* (3)*.*

## Theorem (Lagrange)

*There exists a solution to the Polynomial Interpolation Problem and it is given by*

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x).$$

## Example (Süli and Mayers, E.g., 6.1)

Write down the Lagrange form of the polynomial interpolant to the function $f(x) = e^x$ at interpolation points $\{-1, 0, 1\}$.

# Example 

The figure below shows the solution to Example 9 (top) and the difference between the function $e^x$ and its interpolant (bottom). It would be interesting to see how this error depends on

(i) the function (and its derivatives)

(ii) the number of points used.