§1 Interpolation

§1.2 Finding the interpolant

MA378/531 – Numerical Analysis II ("NA2")

January 2017

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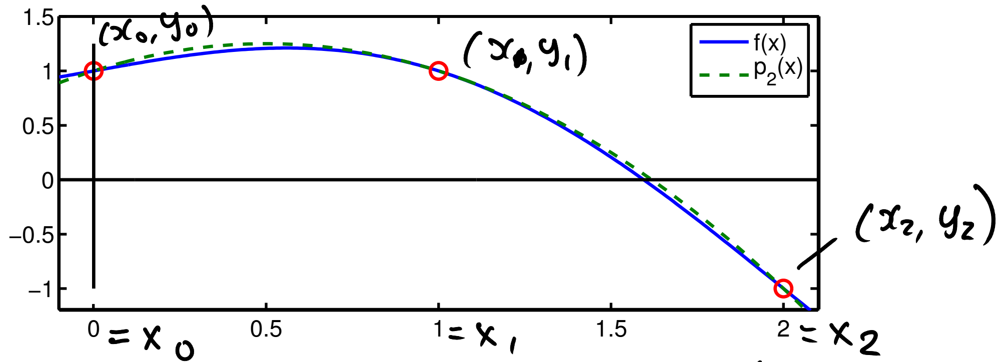
Joseph-Louis Lagrange, born 1736 in Turin, died 1813 in Paris. He made great contributions to many areas of Mathematics, including *Calculus of Variations*.

Example

Show that the polynomial of degree 2 that interpolates $f(x) = 1 - x + \sin(\pi x/2)$ at the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ is $p_2 = -x^2 + x + 1$.



Check: 1. p_2 is quadratic ✓.

$f(x_0) = f(0) = 1 - 0 + 0 = 1$ & $p_2(0) = -0^2 + 0 + 1 = 1$ ✓.

$f(x_1) = f(1) = 1 - 1 + 1 = 1$ & $p_2(1) = -1 + 1 + 1 = 1$ ✓

$f(x_2) = f(2) = 1 - 2 + 0 = -1$ $p_2(2) = -4 + 2 + 1 = -1$ ✓.

It is not hard to convince ourselves that $-x^2 + x + 1$. is the solution to the above PIP. But how to we know we have found the *only* solution? More generally, *under what conditions is there exactly one polynomial that solves the PIP*?

As a first step, we'll prove the following:

Lemma

If $p_n \in \mathcal{P}_n$ has $n+1$ zeros, then $p_n \equiv 0$ (i.e., $p_n(x) = 0$ for all x).

We know that $p_n(x_0) = 0$, $p_n(x_1) = 0$, $\dots$, $p_n(x_n) = 0$

So we can write it as

$$p_n(x) = (x - x_0)(x - x_1)(x - x_2) \dots (x - x_n)\, q(x)$$

for some polynomial $q(x)$. So the coefficient of x^{n+1} is $q(x)$. But $p_n \in \mathcal{P}_n$. So $q(x) \equiv 0$.

So $p_n \equiv 0$.

Theorem (There is a unique solution to the PIP)

There is at most one polynomial of degree $\leq n$ that interpolates the $n+1$ points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) where x_0, x_1, ..., x_n are distinct.

Suppose that $p \in P_n$ & $q \in P_n$ both solve the same PIP. So $p(x_k) = y_k = q(x_k)$ for $k = 0, \dots, n$.

Let $r = p - q$. So $r \in P_n$.

Also $r(x_k) = p(x_k) - q(x_k) = 0$ for $k = 0, \dots, n$.

So, by our previous lemma, $r \equiv 0$.

Thus $p = q$.

Now we want to solve the PIP. It turns out that the most obvious approach may not be the best.

Suppose we are trying to solve the problem as follows: *find p_2 such that*

"find p_2"

$$p_2(x_0) = y_0, \quad p_2(x_1) = y_1, \quad \textit{and} \quad p_2(x_2) = y_2.$$

Since $p_2(x)$ is of the form $a_0 + a_1x + a_2x^2$, this just amounts the finding the values of the coefficients a_0, a_1, and a_2. One might be tempted to solve for them using the system of equations

$$p_2(x_0) = y_0 \quad \Rightarrow \quad \begin{aligned} a_0 + a_1x_0 + a_2x_0^2 &= y_0 \\ a_0 + a_1x_1 + a_2x_1^2 &= y_1 \\ a_0 + a_1x_2 + a_2x_2^2 &= y_2 \end{aligned}$$

This is known as the *Vandermonde System*.

Writing

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 &= y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 &= y_2 \end{aligned}$$

in matrix-vector format we get

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \qquad \text{or} \qquad V\boldsymbol{a} = \boldsymbol{y}. \tag{1}$$

But this may not be a good idea. (*Unfortunately, to see exactly why, you needed to have studied MA385. If you didn't, you can skip the next bit*).

The Vandermonde matrix method

In MA385 we learned about the relationship between the *condition number* of a matrix, V, and the relative error in the (numerical) solution to a matrix-vector equation with V as the coefficient matrix. The condition number is $\kappa(V) = \|V\|\|V^{-1}\|$, for some subordinate matrix norm $\|\cdot\|$.

Example (Stewart's "Afternotes...", Lecture 19)

Suppose $x_0 = 100$, $x_1 = 101$ and $x_2 = 102$. Then it is not hard to check that

$$\|X\|_\infty = \max_i \sum_j |X_{ij}| = 10,507.$$

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Also,

$$V^{-1} = \frac{1}{2}\begin{pmatrix} 10302 & -20400 & 10100 \\ -203 & 404 & -201 \\ 1 & -2 & 1 \end{pmatrix},$$

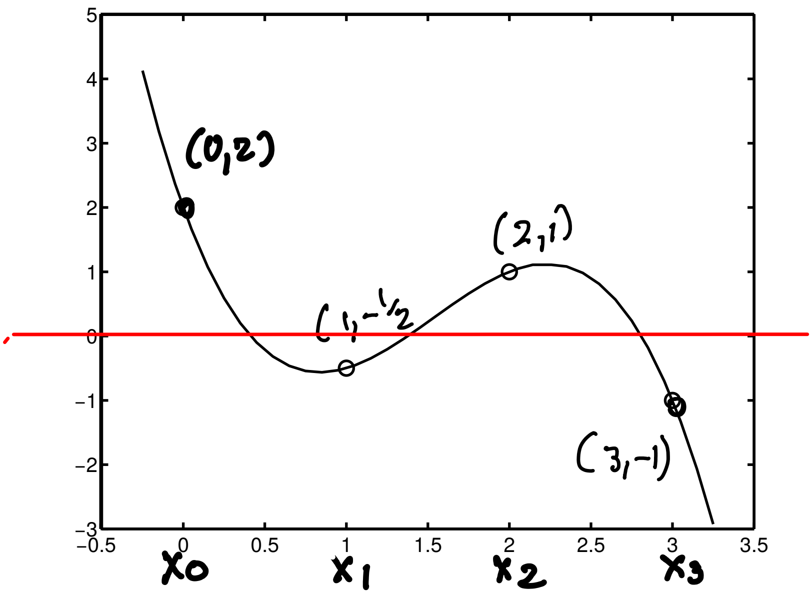
so $\|V^{-1}\|_\infty = 20401$. So $\kappa(V) = 214,353,307$.

We'll now look at a much easier method for solving the Polynomial Interpolation Problem. As a by-product, we get a constructive proof of the existence of a solution to the PIP. (Here "constructive" means that we'll prove it exists by actually computing it).

Example

Consider the problem: *find* $p_3 \in \mathcal{P}_3$ *such that*

$$p_3(0) = 2, \quad p_3(1) = -1/2, \quad p_3(2) = 1, \quad p_3(3) = -1.$$

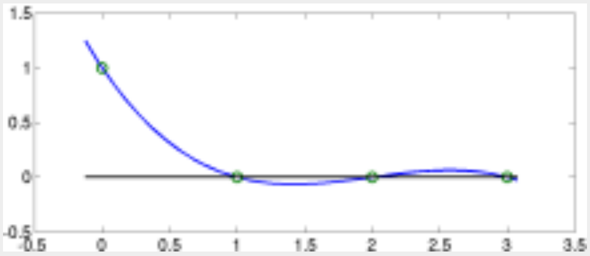


Lagrange Interpolation

Here is an easier problem to solve.

Find $L_0 \in \mathcal{P}_3$ such that

$$L_0(0) = 1, \quad L_0(1) = 0, \quad L_0(2) = 0, \quad L_0(3) = 0.$$

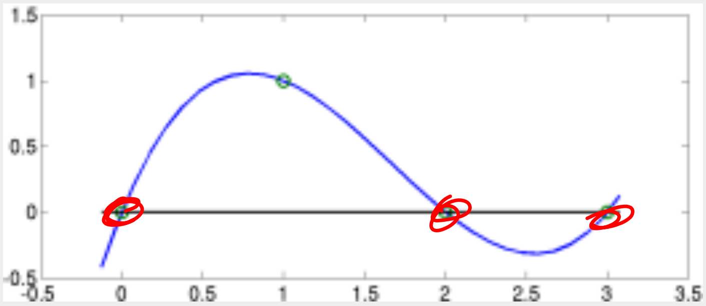


Because L_0 is a cubic and has zeros at $x = 1, 2, 3$ it is of the form $L_0(x) = C(x-1)(x-2)(x-3)$. Choosing C so that $L_0(0) = 1$, we get

$$L_0(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{1}{6}(x-1)(x-2)(x-3).$$

Similarly, let $L_1 \in \mathcal{P}_3$ be the cubic polynomial such that

$$L_1(0) = 0, \quad L_1(1) = 1, \quad L_1(2) = 0, \quad L_1(3) = 0,$$

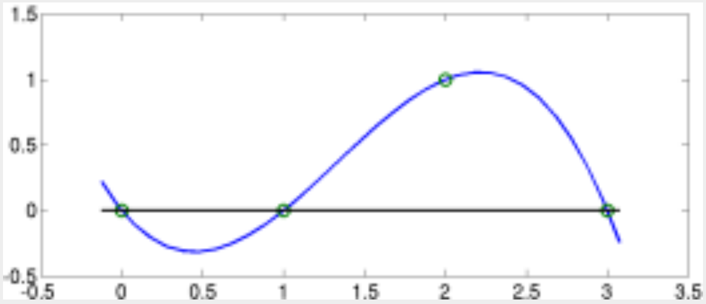


Then

$$L_1(x) = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} =$$

$$\frac{1}{6}(x)(x-2)(x-3).$$

In the same style, let $L_2(x_i) = \begin{cases} 1 & i = 2 \\ 0 & i = 0, 1, 3 \end{cases}$



$$L_2(x) = \frac{(x)(x-1)(x-3)}{(2)(2-1)(2-3)} = -\frac{1}{2}x(x-1)(x-3)$$

Finally, if we define

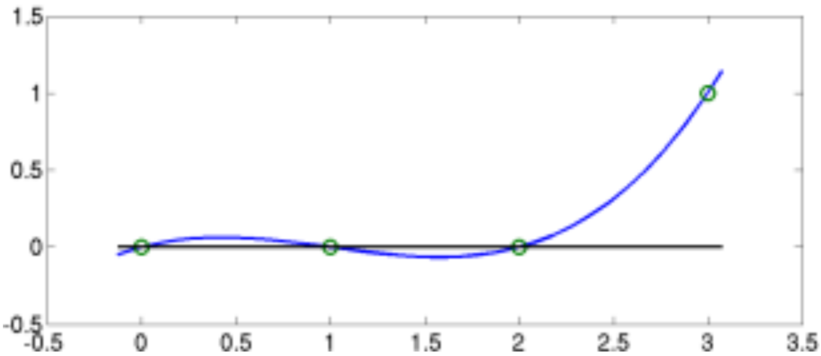
upper-case π (pi).

$$L_3(x_i) = \begin{cases} 1 & i = 3 \\ 0 & i = 0, 1, 2 \end{cases},$$

then clearly,

$=3.$

$$L_3(x) = \frac{(x-0)(x-1)(x-\cancel{3}2)}{(3-0)(3-1)(3-2)} = \prod_{j=0, j\neq 3}^{n} \frac{(x - x_j)}{(x_3 - x_j)}.$$



Lagrange Interpolation

Because each of L_0, L_1, L_2, and L_3 is a cubic, so too is any linear combination of them. So

$$p_3(x) = 2L_0(x) - (1/2)L_1(x) + (1)L_2(x) + (-1)L_3(x),$$

is a cubic. Furthermore

$$\begin{array}{rcllll}
p_3(0) & = & 2L_0(0)^{=1} & -(1/2)L_1(0)^{=0} & +(1)L_2(0)^{=0} & +(-1)L_3(0)^{=0} \\
 & = & 2(1) & -(1/2)(0) & +(1)(0) & +(-1)(0) \\
 & = & 2, & & & \\
p_3(1) & = & 2L_0(1)^{=0} & -(1/2)L_1(1)^{=1} & +(1)L_2(1)^{=0} & +(-1)L_3(1)^{=0} \\
 & = & 2(0) & -(1/2)(1) & +(1)(0) & +(-1)(0) \\
 & = & -1/2, & & & \\
p_3(2) & = & 2L_0(2) & -(1/2)L_1(2) & +(1)L_2(2) & +(-1)L_3(2) \\
 & = & 2(0) & -(1/2)(0) & +(1)(1) & +(-1)(0) \\
 & = & 1, & & & \\
p_3(3) & = & 2L_0(3)^{=0} & -(1/2)L_1(3)^{=0} & +(1)L_2(3)^{=0} & +(-1)L_3(3)^{=1} \\
 & = & 2(0) & -(1/2)(0) & +(1)(0) & +(-1)(1) \\
 & = & -1. & & &
\end{array}$$

Thus p_3 solves the problem!

We can generalise this idea to solve any PIP using what is called *Lagrange* interpolation. We'll now look how to solve the general problem.

Definition (Lagrange Polynomials)

The **Lagrange Polynomials** associated with $x_0 < x_1 < \cdots < x_n$ is the set $\{L_i\}_{i=0}^{n}$ of polynomials in $\mathcal{P}_n$ such that

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{2a}$$

and are given by the formula

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}. \tag{2b}$$

The Lagrange Form

Definition

The **Lagrange form of the Interpolating Polynomial**

PIP 1
$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x), \tag{3a}$$

or

PIP 2.
$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x). \tag{3b}$$

Take care not to confuse the Lagrange Polynomials, which are the L_i with the Lagrange Interpolating Polynomial, which is the p_n defined in (3)*.*

Theorem (Lagrange)

There exists a solution to the Polynomial Interpolation Problem and it is given by

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x).$$

PIP.

Proof. First note that each $L_i \in P_n$. So $p_n \in P_n$.

Also $p_n(x_j) = \sum_{i=0}^{n} y_i L_i(x_j) = y_j$ since

$L_j(x_j) = 1$, and $L_i(x_j) = 0$ when $i \neq j$.

So p_n solves the PIP.

Example (Süli and Mayers, E.g., 6.1)

Write down the Lagrange form of the polynomial interpolant to the function $f(x) = e^x$ at interpolation points $\{-1, 0, 1\}$.

Here $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ & $n = 2$.

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{x(x-1)}{(-1)(-2)} = \tfrac{1}{2}(x)(x-1)$$

$$L_1(x) = 1 - x^2 \qquad L_2(x) = \tfrac{1}{2}(x)(x+1)$$

So

$$p_2(x) = e^{-1} L_0(x) + e^0 L_1(x) + e^1 L_2(x)$$

$$= \tfrac{1}{2} e^{-1} x(x-1) + (1 - x^2) + \tfrac{1}{2} e\, x(x+1).$$

$$\left(\ldots = 1 + x\left(\tfrac{1}{2}e - \tfrac{1}{2}e^{-1}\right) + x^2\left(\tfrac{1}{2}e + \tfrac{1}{2}e^{-1} - 1\right). \right)$$

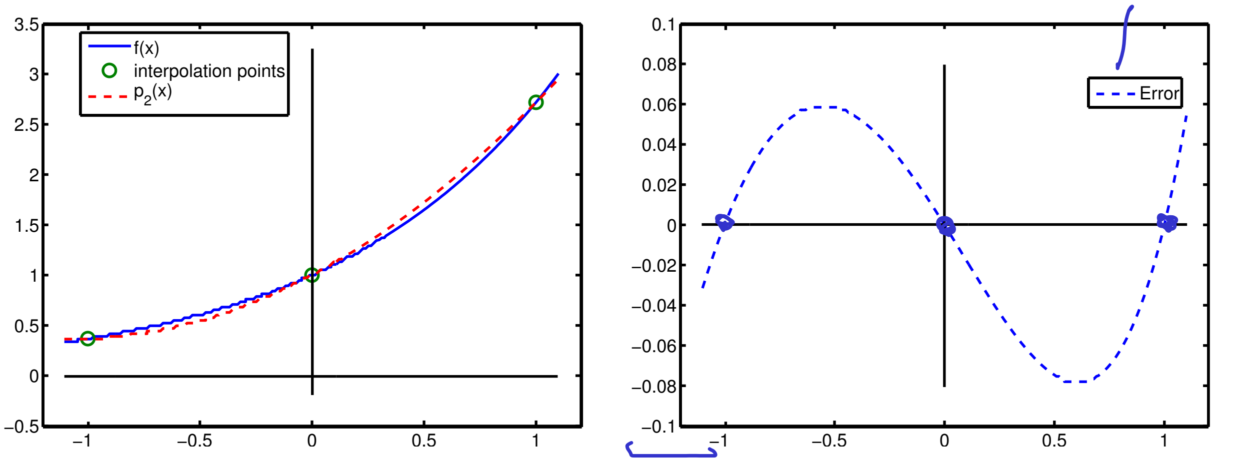
Example

The figure below shows the solution to Example 9 (~~top~~ left) and the difference between the function e^x and its interpolant (~~bottom~~ Right). It would be interesting to see how this error depends on

(i) the function (and its derivatives)

(ii) the number of points used.

$f(x) - p_2(x)$



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