

## §1 Interpolation

# §1.3 Interpolation Error Estimates

MA378/531 – Numerical Analysis II (“NA2”)

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Augustin-Louis Cauchy (1789–1857), Paris, France. He was a pioneer of analysis, in particular in introducing rigour into calculus proofs. He founded the fields of complex analysis and the study of permutation groups.

In our last example, we wrote down the polynomial of degree  $n = 2$  interpolating  $f(x) = e^x$  at  $x_0 = -1$ ,  $x_1 = 0$  and  $x_2 = 1$ .

We now want to investigate how, in general, error in polynomial interpolation depends on

- (i) the function (and its derivatives)
- (ii) the number of points used (or, equivalently, degree of the polynomial used).

The main ingredient we need to the following theorem.

### Theorem (Rolle's Theorem)

*Let  $g$  be a function that is continuous and differentiable on the interval  $[a, b]$ . If  $g(a) = g(b)$ , then there is at least one point  $c$  in  $(a, b)$  where  $g'(c) = 0$ .*

Our “proof” is by picture:<sup>1</sup>

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<sup>1</sup>One can easily deduce Rolle's Theorem from the Mean Value Theorem (MVT). But since the standard proof of the MVT uses Rolle's Theorem, that would be cheating

The simplest case is when  $n = 0$ , so the interpolant is a constant, i.e., it is  $p_0$  interpolating a function  $f$  at a point  $x_0$ . Here is one way we can deduce the *interpolation error*.

It is important to understand what this formula is telling us:

We will use this Rolle's Theorem to prove the most important theorem of NA2; it is used repeatedly through-out the course. It's often called the *Polynomial Interpolation Error Theorem*, but we'll call it *Cauchy's Theorem* for short.

First, we need to define an important polynomial.

### Definition (Nodal Polynomial)

The **Nodal Polynomial**  $\pi_{n+1}$  associated with the interpolation points that  $a = x_0 < x_1 < \cdots < x_n = b$  is

$$\pi_{n+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_n) = \prod_{i=0}^n (x - x_i).$$

**Theorem (Cauchy, 1840)**

*Suppose that  $n \geq 0$  and  $f$  is a real-valued function that is continuous and defined on  $[a, b]$ , such that the derivative of  $f$  of order  $n + 1$  exists and is continuous on  $[a, b]$ . The  $p_n$  be the polynomial of degree  $n$  that interpolates  $f$  at the  $n + 1$  points  $a = x_0 < x_1 < \cdots < x_n = b$ . Then, for any  $x \in [a, b]$  there is a  $\tau \in (a, b)$  such that*

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{(n+1)!} \pi_{n+1}(x). \quad (1)$$







**Example**

In an earlier example, we wrote down the Lagrange form of the polynomial,  $p_2$ , that interpolates  $f(x) = e^x$  at the points  $\{-1, 0, 1\}$ . Give a formula for  $e^x - p_2(x)$ .

Usually (and as in the above example), we can't calculate  $f(x) - p_n(x)$  exactly from Formula (1), because we have no way of finding  $\tau$ . However, we are typically not so interested in what the error is at some given point, but what is the maximum error over the whole interval  $[x_0, x_n]$ . That is given by:

### Corollary

*Define*

$$M_{n+1} = \max_{\sigma \in [x_0, x_n]} |f^{(n+1)}(\sigma)|.$$

*Then*

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|. \quad (2)$$

**Example**

Let  $p_1$  be the polynomial of degree 1 that interpolates a function  $f$  at distinct points  $x_0$  and  $x_1$ . Letting  $h = x_1 - x_0$ , show that

$$\max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{1}{8} h^2 M_2.$$